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A Generalization of Poincaré's Theorem to Hybrid and Impulsive Dynamical Systems

by

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Abstract

Poincaré's method is well known for analyzing the stability of continuous-time dynamical systems with periodic solutions by studying the stability properties of a fixed point as an equilibrium point of a discrete-time system. In this paper we generalize Poincaré's method to dynamical systems possessing left-continuous flows to address the stability of limit cycles and periodic orbits of left-continuous, hybrid, and impulsive dynamical systems. It is shown that the resetting manifold (which gives rise to state discontinuities) provides a natural hyperplane for defining a Poincaré return map. In the special case of impulsive dynamical systems, we show that the Poincaré map replaces an n th-order impulsive dynamical system by an $(n - 1)$ th-order discrete-time system for analyzing the stability of periodic orbits.

Key Words: Hybrid dynamical systems, periodic orbits, limit cycles, Poincaré maps, stability of periodic orbits

Running Title: Hybrid Dynamical Systems with Periodic Solutions

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1. Introduction

In certain dynamical systems and in particular mechanical and biological systems, system state discontinuities arise naturally. In a recent series of papers [1, 2] a unified dynamical systems framework for a general class of systems possessing left-continuous flows; that is, left-continuous dynamical systems, was developed. A left-continuous dynamical system is a precise mathematical object that is defined on the semi-infinite interval as a mapping between vector spaces satisfying an appropriate set of axioms and includes hybrid [1–4] and impulsive dynamical systems [5–10] as special cases. Stability analysis of left-continuous dynamical systems is also considered in [1, 2], with [2] presenting invariant set stability theorems for a class of left-continuous and impulsive dynamical systems. The extension of the invariance principle to impulsive dynamical systems presented in [2] provides a powerful tool in analyzing the stability properties of periodic orbits and limit cycles of dynamical systems with impulse effects. However, the periodic orbit of a left-continuous dynamical system is a disconnected set in the n -dimensional state space making the construction of a Lyapunov-like function satisfying the invariance principle a daunting task for high-order nonlinear systems. In such cases, it becomes necessary to seek alternative tools to study the stability of periodic orbits of hybrid and impulsive dynamical systems, especially if the trajectory of the system can be relatively easily integrated.

In this paper we generalize Poincaré’s theorem to left-continuous dynamical systems and hence to hybrid and impulsive dynamical systems. Specifically, we develop necessary and sufficient conditions for stability of periodic orbits based on the stability properties of a fixed point of a discrete-time dynamical system constructed from a Poincaré return map. As opposed to dynamical systems possessing continuous flows requiring the construction of a hyperplane that is transversal to a candidate periodic trajectory necessary for defining the return map, the resetting set which provides a criterion for determining when the states of the left-continuous dynamical system are to be reset provides a natural candidate for the transversal surface on which the Poincaré map of a left-continuous dynamical system can be defined. Hence, the Poincaré return map is defined by a subset of the resetting set that induces a discrete-time mapping from this subset onto the resetting set. This mapping

traces the left-continuous trajectory of the left-continuous dynamical system from a point on the resetting set to its next corresponding intersection with the resetting set. In the case of impulsive dynamical systems possessing sufficiently smooth resetting manifolds, we show the Poincaré return map can be used to establish a relationship between the stability properties of an impulsive dynamical system with periodic solutions and the stability properties of an equilibrium point of an $(n - 1)$ th-order discrete-time system. These results have been recently employed to analyze the periodic orbits for the verge and folio clock escapement mechanism [11] which exhibits impulsive dynamics. Finally, we note that a related approach to the present paper restricted to impulsive dynamical systems with the additional restriction that the periodic trajectory resets at the same point on the resetting set is addressed in [12].

2. Left-Continuous Dynamical Systems with Periodic Solutions

In this section we establish definitions, notation, and provide several key results on left-continuous dynamical systems with periodic solutions. Let \mathbb{R} denote the set of real numbers, let \mathbb{R}^n denote the set of $n \times 1$ real column vectors, and let \mathcal{N} denote the set of nonnegative integers. We write $\|\cdot\|$ for the Euclidean vector norm, $\mathcal{B}_\epsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\epsilon > 0$, for the open ball centered at α with radius ϵ , and $\text{dist}(p, \mathcal{M})$ for the smallest distance from a point p to any point in the set \mathcal{M} ; that is, $\text{dist}(p, \mathcal{M}) \triangleq \inf_{x \in \mathcal{M}} \|p - x\|$. The following definition is concerned with left-continuous dynamical systems; that is, systems with left-continuous flows [1, 2]. For this definition $\mathcal{D} \subseteq \mathbb{R}^n$ and $\mathcal{T}_{x_0} \subseteq [0, \infty)$, $x_0 \in \mathcal{D}$, is a dense subset of the semi-infinite interval $[0, \infty)$ such that $[0, \infty) \setminus \mathcal{T}_{x_0}$ is (finitely or infinitely) countable.

Definition 2.1 [1, 2]. A *left-continuous dynamical system*¹ on \mathcal{D} is the triple $(\mathcal{D}, [0, \infty), s)$, where $s : [0, \infty) \times \mathcal{D} \rightarrow \mathcal{D}$ is such that the following axioms hold:

i) (Left-continuity): $s(\cdot, x_0)$ is left-continuous in t ; that is, $\lim_{\tau \rightarrow t^-} s(\tau, x_0) = s(t, x_0)$ for all $x_0 \in \mathcal{D}$ and $t \in (0, \infty)$.

ii) (Consistency): $s(0, x_0) = x_0$ for all $x_0 \in \mathcal{D}$.

iii) (Semi-group property): $s(\tau, s(t, x_0)) = s(t + \tau, x_0)$ for all $x_0 \in \mathcal{D}$ and $t, \tau \in [0, \infty)$.

¹Right-continuous dynamical systems; that is, systems possessing right-continuous flows, can also be analogously considered here.

iv) (Quasi-continuous dependence): For every $x_0 \in \mathcal{D}$, there exists $\mathcal{T}_{x_0} \subseteq [0, \infty)$ such that $[0, \infty) \setminus \mathcal{T}_{x_0}$ is countable and for every $\varepsilon > 0$ and $t \in \mathcal{T}_{x_0}$, there exists $\delta(\varepsilon, x_0, t) > 0$ such that if $\|x_0 - y\| < \delta(\varepsilon, x_0, t)$, $y \in \mathcal{D}$, then $\|s(t, x_0) - s(t, y)\| < \varepsilon$.

Henceforth, we denote the left-continuous dynamical system $(\mathcal{D}, [0, \infty), s)$ by \mathcal{G} . Furthermore, we refer to $s(t, x_0)$, $t \geq 0$, as the *trajectory* of \mathcal{G} corresponding to $x_0 \in \mathcal{D}$, and for a given trajectory $s(t, x_0)$, $t \geq 0$, we refer to $x_0 \in \mathcal{D}$ as an *initial condition* of \mathcal{G} . The trajectory $s(t, x_0)$, $t \geq 0$, of \mathcal{G} is *bounded* if there exists $\gamma > 0$ such that $\|s(t, x_0)\| < \gamma$, $t \geq 0$. Finally, for the remainder of the paper we refer to the left-continuous dynamical system \mathcal{G} as the dynamical system \mathcal{G} .

The quasi-continuous dependence property *iv)* is a generalization of the standard continuous dependence property for dynamical systems with continuous flows to dynamical systems with left-continuous flows. Specifically, by letting $\mathcal{T}_{x_0} = \overline{\mathcal{T}_{x_0}} = [0, \infty)$, where $\overline{\mathcal{T}_{x_0}}$ denotes the closure of the set \mathcal{T}_{x_0} , the quasi-continuous dependence property specializes to the classical continuous dependence of solutions of a given dynamical system with respect to the system's initial conditions $x_0 \in \mathcal{D}$ [13]. If, in addition, $x_0 = 0$, $s(t, 0) = 0$, $t \geq 0$, and $\delta(\varepsilon, 0, t)$ can be chosen independent of t , then continuous dependence implies the classical Lyapunov stability of the zero trajectory $s(t, 0) = 0$, $t \geq 0$. Hence, Lyapunov stability of motion can be interpreted as continuous dependence of solutions uniformly in t for all $t \geq 0$. Conversely, continuous dependence of solutions can be interpreted as Lyapunov stability of motion for every fixed time t [13]. Analogously, Lyapunov stability of impulsive dynamical systems as defined in [5] can be interpreted as quasi-continuous dependence of solutions uniformly in t for all $t \in \mathcal{T}_{x_0}$. The following proposition provides sufficient conditions for \mathcal{G} to be a left-continuous dynamical system.

Proposition 2.1 [2]. Let the triple $(\mathcal{D}, [0, \infty), s)$, where $s : [0, \infty) \times \mathcal{D} \rightarrow \mathcal{D}$, be such that Axioms *i)–iii)* hold and:

iv)' For every $x_0 \in \mathcal{D}$, $\varepsilon, \eta > 0$, and $T \in \mathcal{T}_{x_0}$, there exists $\delta(\varepsilon, \eta, x_0, T) > 0$ such that if $\|x_0 - y\| < \delta(\varepsilon, \eta, x_0, T)$, $y \in \mathcal{D}$, then for every $t \in \mathcal{T}_{x_0} \cap [0, T]$ such that $|t - \tau| > \eta$, for all $\tau \in [0, T] \setminus \mathcal{T}_{x_0}$, $\|s(t, x_0) - s(t, y)\| < \varepsilon$. Furthermore, if $t \in \mathcal{T}_{x_0}$ is an accumulation point of $[0, \infty) \setminus \mathcal{T}_{x_0}$, then $s(t, \cdot)$ is continuous at x_0 .

Then \mathcal{G} is a left-continuous dynamical system.

Definition 2.2 [1]. A *strong left-continuous dynamical system* on \mathcal{D} is a left-continuous dynamical system on \mathcal{D} and the triple $(\mathcal{D}, [0, \infty), s)$ is such that Axiom *iv)*' holds.

Henceforth, we assume that \mathcal{T}_{x_0} in Definition 2.1 is given by $\mathcal{T}_{x_0} \triangleq \{t \in [0, \infty) : s(t, x_0) = s(t^+, x_0)\}$ so that $[0, \infty) \setminus \mathcal{T}_{x_0}$ corresponds to the (countable) set of *resetting times*, where the trajectory $s(\cdot, x_0)$ is discontinuous. Next, define

$$\mathcal{S}_{x_0} \triangleq \{x \in \mathbb{R}^n : \text{there exists } t \in [0, \infty) \setminus \mathcal{T}_{x_0} \text{ such that } x = s(t, x_0)\} = s([0, \infty) \setminus \mathcal{T}_{x_0}, x_0),$$

and $\mathcal{S} \triangleq \cup_{x_0 \in \mathcal{D}} \mathcal{S}_{x_0}$. Henceforth, \mathcal{S} is called the *resetting set*. Furthermore, let $\tau_i(x_0)$, $i \in \mathcal{N}$, where $\tau_0(x_0) \triangleq 0$ and $\tau_1(x_0) < \tau_2(x_0) < \dots$, denote the resetting times; that is, $\{\tau_1(x_0), \tau_2(x_0), \dots\} = [0, \infty) \setminus \mathcal{T}_{x_0}$. Next, we present a key assumption on the resetting times $\tau_i(\cdot)$, $i \in \mathcal{N}$.

Assumption 2.1. For every $i \in \mathcal{N}$, $\tau_i(\cdot)$ is continuous and for every $x_0 \in \mathcal{D}$, there exists $\varepsilon(x_0)$ such that $\tau_{i+1}(x_0) - \tau_i(x_0) \geq \varepsilon(x_0)$, $i \in \mathcal{N}$.

The next result shows that \mathcal{G} is a strong left-continuous dynamical system if and only if the trajectory of \mathcal{G} is *jointly continuous between resetting events*; that is, for every $\epsilon > 0$ and $k \in \mathcal{N}$ there exists $\delta = \delta(\epsilon, k) > 0$ such that if $|t - t'| + \|x_0 - y\| < \delta$, then $\|s(t, x_0) - s(t', y)\| < \epsilon$, where $x_0, y \in \mathcal{D}$, $t \in (\tau_k(x_0), \tau_{k+1}(x_0)]$, and $t' \in (\tau_k(y), \tau_{k+1}(y)]$.

Proposition 2.2. Consider the dynamical system \mathcal{G} satisfying Axioms *i) – iii)* and Assumption 2.1. Then \mathcal{G} is a strong left-continuous dynamical system if and only if the trajectory $s(t, x_0)$, $t \geq 0$, of \mathcal{G} is jointly continuous between resetting events.

Proof. Assume \mathcal{G} is a strong left-continuous dynamical system, let $\epsilon > 0$, and let $k \in \mathcal{N}$. Since, by Assumption 2.1, $\tau_k(\cdot)$ is continuous, it follows that for sufficiently small $\delta_1 > 0$, $\tau_k(x)$ and $\tau_{k+1}(x)$, $x \in \mathcal{B}_{\delta_1}(x_0)$, are well-defined and finite. Hence, it follows from the strong quasi-continuity property *iv)*' that $s(t, \cdot)$, $t \in (\tau_k(\cdot), \tau_{k+1}(\cdot)]$, is uniformly bounded on $\mathcal{B}_{\delta_1}(x_0)$. Now, since \mathcal{G} is left-continuous it follows that for $\epsilon > 0$ and $k \in \mathcal{N}$ there exists $\hat{\delta} = \hat{\delta}(\epsilon, k) > 0$ such that if $|t - t'| < \hat{\delta}$, then

$$\|s(t, x) - s(t', x)\| < \frac{\epsilon}{3}, \quad x \in \mathcal{B}_{\delta_1}(x_0), \quad t, t' \in (\tau_k(x), \tau_{k+1}(x)]. \quad (1)$$

Next, it follows from Assumption 2.1 that for every sufficiently small $\lambda > 0$ and $k \in \mathcal{N}$, $\underline{\tau}_k(\lambda, x_0) \triangleq \inf_{x \in \mathcal{B}_\lambda(x_0)} \tau_k(x)$ and $\bar{\tau}_k(\lambda, x_0) \triangleq \sup_{x \in \mathcal{B}_\lambda(x_0)} \tau_k(x)$ are well defined and $\lim_{\lambda \rightarrow 0} \underline{\tau}_k(\lambda, x_0) = \lim_{\lambda \rightarrow 0} \bar{\tau}_k(\lambda, x_0) = \tau_k(x_0)$. (Note that $\underline{\tau}_k(\lambda, x_0) \leq \tau_k(x) \leq \bar{\tau}_k(\lambda, x_0)$, for all $x \in \mathcal{B}_\lambda(x_0)$.) Hence, there exists $\delta' = \delta'(\hat{\delta}) > 0$ such that $\bar{\tau}_k(\delta', x_0) - \underline{\tau}_k(\delta', x_0) < \hat{\delta}$ and $\bar{\tau}_{k+1}(\delta', x_0) - \underline{\tau}_{k+1}(\delta', x_0) < \hat{\delta}$. Next, let $\eta > 0$ be such that

$$\bar{\tau}_k(\delta', x_0) - \underline{\tau}_k(\delta', x_0) < \tau_k(x_0) - \underline{\tau}_k(\delta', x_0) + \eta < \hat{\delta}, \quad (2)$$

$$\bar{\tau}_{k+1}(\delta', x_0) - \underline{\tau}_{k+1}(\delta', x_0) < \eta + \bar{\tau}_{k+1}(\delta', x_0) - \tau_{k+1}(x_0) < \hat{\delta}. \quad (3)$$

Then, it follows from the strong quasi-continuity of \mathcal{G} that there exists $\delta'' = \delta''(\epsilon, \eta, k) > 0$ such that

$$\|s(t, x_0) - s(t, y)\| < \frac{\epsilon}{3}, \quad y \in \mathcal{B}_{\delta''}(x_0), \quad t \in (\tau_k(x_0) + \eta, \tau_{k+1}(x_0) - \eta). \quad (4)$$

Now, if $|t - t'| + \|x_0 - y\| < \delta$, where $\delta = \min\{\delta_1, \delta', \delta'', \hat{\delta}\}$, $t \in (\tau_k(x_0) + \eta, \tau_{k+1}(x_0) - \eta)$, and $t' \in (\tau_k(y), \tau_{k+1}(y)]$, then it follows from (1), (4), and triangular inequality for vector norms that

$$\|s(t, x_0) - s(t', y)\| \leq \|s(t, x_0) - s(t, y)\| + \|s(t, y) - s(t', y)\| < \frac{2}{3}\epsilon < \epsilon. \quad (5)$$

Finally, if $|t - t'| + \|x_0 - y\| < \delta$, where $t \in (\tau_k(x_0), \tau_{k+1}(x_0)] \setminus (\tau_k(x_0) + \eta, \tau_{k+1}(x_0) - \eta)$ and $t' \in (\tau_k(y), \tau_{k+1}(y)]$, then conditions (2), (3) imply that there exists $t'' \in (\tau_k(x_0) + \eta, \tau_{k+1}(x_0) - \eta)$ such that $|t - t''| < \hat{\delta}$ and $|t' - t''| < \hat{\delta}$. Hence, by (1) and (4) it follows that

$$\|s(t, x_0) - s(t', y)\| \leq \|s(t, x_0) - s(t'', x_0)\| + \|s(t'', x_0) - s(t'', y)\| + \|s(t'', y) - s(t', y)\| < \epsilon,$$

which establishes that \mathcal{G} is jointly continuous between resetting events.

To show that joint continuity of $s(t, x_0)$, $t \geq 0$, between resetting events implies strong left-continuity of \mathcal{G} , let $\epsilon, \eta > 0$, $T \in \mathcal{T}_{x_0}$, and suppose $\tau_k(x_0) < T < \tau_{k+1}(x_0)$. Then, it follows from the joint continuity of \mathcal{G} that there exists $\delta' = \delta'(\epsilon, k) > 0$ such that if $|t - t'| + \|x_0 - y\| < \delta'$, then $\|s(t, x_0) - s(t', y)\| < \epsilon$, where $x_0, y \in \mathcal{D}$, $t \in (\tau_k(x_0), \tau_{k+1}(x_0)]$, and $t' \in (\tau_k(y), \tau_{k+1}(y)]$. Now, it follows from the Assumption 2.1 that there exists $\delta'' = \delta''(x_0, \eta, k) > 0$ such that $\bar{\tau}_k(\delta'', x_0) - \tau_k(x_0) < \eta$ and $\tau_{k+1}(x_0) - \underline{\tau}_{k+1}(\delta'', x_0) < \eta$. Note that the above inequalities guarantee that if $t = t' \in (\tau_k(x_0) + \eta, \tau_{k+1}(x_0) - \eta)$, then $t \in (\tau_k(x_0), \tau_{k+1}(x_0)]$

and $t' \in (\tau_k(y), \tau_{k+1}(y)]$, $y \in \mathcal{B}_{\delta'}(x_0)$. Furthermore, letting $\delta_k = \delta_k(\epsilon, \eta, x_0, k) = \min\{\delta', \delta''\}$, it follows from the joint continuity of \mathcal{G} that for $t = t' \in (\tau_k(x_0) + \eta, \tau_{k+1}(x_0) - \eta)$, $\|s(t, x_0) - s(t, y)\| < \epsilon$, $y \in \mathcal{B}_{\delta_k}(x_0)$. Similarly, we can obtain $\delta_{k-1} = \delta_{k-1}(\epsilon, \eta, x_0, k) > 0$ such that an analogous inequality can be constructed for all $y \in \mathcal{B}_{\delta_{k-1}}(x_0)$ and $t \in (\tau_{k-1}(x_0) + \eta, \tau_k(x_0) - \eta)$. Recursively repeating this procedure for $m = k - 2, \dots, 1$, and choosing $\delta = \delta(\epsilon, \eta, x_0, k) = \delta(\epsilon, \eta, x_0, T) = \min\{\delta_1, \dots, \delta_k\}$, it follows that $\|s(t, x_0) - s(t, y)\| < \epsilon$, $y \in \mathcal{B}_\delta(x_0)$, $t \in [0, T]$, and $|t - \tau_l(x_0)| > \eta$, $l = 1, \dots, k$, which implies that \mathcal{G} is a strong left-continuous dynamical system. \square

Definition 2.3. A solution $s(t, x_0)$ of \mathcal{G} is *periodic* if there exists a finite time $T > 0$, known as the period, such that $s(t+T, x_0) = s(t, x_0)$ for all $t \geq 0$. A set $\mathcal{O} \subset \mathcal{D}$ is a *periodic orbit* of \mathcal{G} if $\mathcal{O} = \{x \in \mathcal{D} : x = s(t, x_0), 0 \leq t \leq T\}$ for some periodic solution $s(t, x_0)$ of \mathcal{G} .

Note that the set \mathcal{S}_{x_0} is identical for all $x_0 \in \mathcal{O}$. Furthermore, if for every $x \in \mathbb{R}^n$ there exists $\varepsilon(x)$ such that $\tau_{i+1}(x) - \tau_i(x) \geq \varepsilon(x)$, $i \in \mathcal{N}$, then it follows that \mathcal{S}_x contains a finite number of (isolated) points. Finally, for every $x_0 \in \mathcal{O}$ it follows that $\tau_{i+N}(x_0) = \tau_i(x_0) + T$, $i = 2, 3, \dots$, where N denotes the number of points in \mathcal{S}_{x_0} .

Next, to extend Poincaré's theorem to hybrid dynamical systems let $\hat{\mathcal{S}} \subset \mathcal{S}$ be such that $\mathcal{O} \cap \hat{\mathcal{S}}$ is a singleton. Note that the existence of such an $\hat{\mathcal{S}}$ is guaranteed since all the points in $\mathcal{O} \cap \mathcal{S}$ are isolated. Now, we define the Poincaré return map $P : \hat{\mathcal{S}} \rightarrow \mathcal{S}$ by

$$P(x) \triangleq s(\tau_{N+1}(x), x), \quad x \in \hat{\mathcal{S}}. \quad (6)$$

Note that if $p \in \mathcal{O} \cap \hat{\mathcal{S}}$, then $s(\tau_{N+1}(p), p) = p$. Furthermore, if Assumption 2.1 holds then $\tau_{N+1}(\cdot)$ is continuous and hence it follows that $P(\cdot)$ is well defined. Next, define the discrete-time system given by

$$z(k+1) = P(z(k)), \quad k \in \mathcal{N}, \quad z(0) \in \hat{\mathcal{S}}. \quad (7)$$

It is easy to see that p is a fixed point of (7). For notational convenience define the set $\Theta_{x_0, \eta} \triangleq \{t \in \mathcal{T}_{x_0} : |t - \tau| > \eta, \tau \in [0, \infty) \setminus \mathcal{T}_{x_0}\}$ denoting the set of all non-resetting times that are at least a distance η away from the resetting times.

Next, we introduce the notions of Lyapunov and asymptotic stability of a periodic orbit for the left-continuous dynamical system \mathcal{G} .

Definition 2.4. A periodic orbit \mathcal{O} of \mathcal{G} is *Lyapunov stable* if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $\text{dist}(x_0, \mathcal{O}) < \delta$, then $\text{dist}(s(t, x_0), \mathcal{O}) < \epsilon$, $t \geq 0$.

Definition 2.5. A periodic orbit \mathcal{O} of \mathcal{G} is *asymptotically stable* if it is Lyapunov stable and there exists $\delta > 0$ such that if $\text{dist}(x_0, \mathcal{O}) < \delta$, then $\text{dist}(s(t, x_0), \mathcal{O}) \rightarrow 0$ as $t \rightarrow \infty$.

The following key lemma is needed for the main stability result of this section.

Lemma 2.1. Consider the strong left-continuous dynamical system \mathcal{G} . Assume the point $p \in \hat{\mathcal{S}}$ generates the periodic orbit $\mathcal{O} \triangleq \{x \in \mathcal{D} : x = s(t, p), 0 \leq t \leq T\}$, where $s(t, p), t \geq 0$, is the periodic solution with the period $T = \tau_{N+1}(p)$. Then the following statements hold:

i) The periodic orbit \mathcal{O} is Lyapunov stable if and only if for every $\epsilon > 0$ and for every $p_{\mathcal{O}} \in \overline{\mathcal{O}}$ there exists $\delta' = \delta'(\epsilon, p_{\mathcal{O}}) > 0$ such that if $x_0 \in \mathcal{B}_{\delta'}(p_{\mathcal{O}})$, then $\text{dist}(s(t, x_0), \mathcal{O}) < \epsilon$, $t \geq 0$.

ii) The periodic orbit \mathcal{O} is asymptotically stable if and only if it is Lyapunov stable and for every $p_{\mathcal{O}} \in \overline{\mathcal{O}}$ there exists $\delta' = \delta'(p_{\mathcal{O}}) > 0$ such that if $x_0 \in \mathcal{B}_{\delta'}(p_{\mathcal{O}})$, then $\text{dist}(s(t, x_0), \mathcal{O}) \rightarrow 0$ as $t \rightarrow \infty$.

Proof. *i)* Necessity is immediate. To show sufficiency, assume that for every $\epsilon > 0$ and for every $p_{\mathcal{O}} \in \overline{\mathcal{O}}$ there exists $\delta' = \delta'(\epsilon, p_{\mathcal{O}}) > 0$ such that if $x_0 \in \mathcal{B}_{\delta'}(p_{\mathcal{O}})$, then $\text{dist}(s(t, x_0), \mathcal{O}) < \epsilon$, $t \geq 0$. Here, we assume that $\delta' = \delta'(\epsilon, p_{\mathcal{O}}) > 0$ is the largest value such that the above distance inequality holds. Next, let $\delta = \delta(\epsilon) = \inf_{p_{\mathcal{O}} \in \overline{\mathcal{O}}} \delta'(\epsilon, p_{\mathcal{O}})$ and suppose, *ad absurdum*, that $\delta = 0$. In this case, there exists a sequence $\{p_{\mathcal{O}k}\}_{k=1}^{\infty} \in \overline{\mathcal{O}}$ such that $\lim_{k \rightarrow \infty} \delta'(\epsilon, p_{\mathcal{O}k}) = 0$. Since $\{p_{\mathcal{O}k}\}_{k=1}^{\infty}$ is a bounded sequence, it follows from the Bolzano-Weierstrass theorem [14] that there exists a convergent subsequence $\{q_{\mathcal{O}k}\}_{k=1}^{\infty} \in \{p_{\mathcal{O}k}\}_{k=1}^{\infty}$ such that $\lim_{k \rightarrow \infty} q_{\mathcal{O}k} = q$ and $\lim_{k \rightarrow \infty} \delta'(\epsilon, q_{\mathcal{O}k}) = 0$. Note, that since $\overline{\mathcal{O}}$ is closed and $\{q_{\mathcal{O}k}\}_{k=1}^{\infty} \in \overline{\mathcal{O}}$, it follows that $q \in \overline{\mathcal{O}}$ and hence $\delta'(\epsilon, q) > 0$. Thus, it follows that there exists $\tilde{q} \in \{q_{\mathcal{O}k}\}_{k=1}^{\infty}$ such that, for sufficiently small $\mu > 0$, $\mathcal{B}_{\delta'(\epsilon, \tilde{q}) + \mu}(\tilde{q}) \subset \mathcal{B}_{\delta'(\epsilon, q)}(q)$. Now, since, for every $p_{\mathcal{O}} \in \overline{\mathcal{O}}$, $\delta' = \delta'(\epsilon, p_{\mathcal{O}}) > 0$ is assumed to be the largest value such that $\text{dist}(s(t, x_0), \mathcal{O}) < \epsilon$, $t \geq 0$, for all $x_0 \in \mathcal{B}_{\delta'}(p_{\mathcal{O}})$, it follows that there exists $x'_0 \in \mathcal{B}_{\delta'(\epsilon, \tilde{q}) + \mu}(\tilde{q})$ and $t' \geq 0$ such that $\text{dist}(s(t', x'_0), \mathcal{O}) > \epsilon$. However, since $\mathcal{B}_{\delta'(\epsilon, \tilde{q}) + \mu}(\tilde{q}) \subset \mathcal{B}_{\delta'(\epsilon, q)}(q)$, then for $x'_0 \in \mathcal{B}_{\delta'(\epsilon, \tilde{q}) + \mu}(\tilde{q})$ it follows that $\text{dist}(s(t, x'_0), \mathcal{O}) < \epsilon$ for all $t \geq 0$, which is a contradiction.

Hence, for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for every $p_{\mathcal{O}} \in \overline{\mathcal{O}}$ and $x_0 \in \mathcal{B}_\delta(p_{\mathcal{O}})$ it follows that $\text{dist}(s(t, x_0), \mathcal{O}) < \epsilon, t \geq 0$. Next, given $x_0 \in \mathcal{D}$ such that $\text{dist}(x_0, \mathcal{O}) = \inf_{p_{\mathcal{O}} \in \mathcal{O}} \|x_0 - p_{\mathcal{O}}\| < \delta$, it follows that there exist a point $p^* \in \mathcal{O}$ such that $\text{dist}(x_0, \mathcal{O}) \leq \|x_0 - p^*\| < \delta$, which implies that $x_0 \in \mathcal{B}_\delta(p^*)$ and hence $\text{dist}(s(t, x_0), \mathcal{O}) < \epsilon, t \geq 0$, establishing Lyapunov stability.

ii) The proof is analogous to *i)* and hence is omitted. □

The following theorem generalizes Poincaré's theorem to strong left-continuous dynamical systems by establishing a relationship between the stability properties of the periodic orbit \mathcal{O} and the stability properties of an equilibrium point of the discrete-time system (7).

Theorem 2.1. Consider the strong left-continuous dynamical system \mathcal{G} with the Poincaré return map defined by (6). Assume that Assumption 2.1 holds and the point $p \in \hat{\mathcal{S}}$ generates the periodic orbit $\mathcal{O} \triangleq \{x \in \mathcal{D} : x = s(t, p), 0 \leq t \leq T\}$, where $s(t, p), t \geq 0$, is the periodic solution with the period $T = \tau_{N+1}(p)$ such that $s(\tau_{N+1}(p), p) = p$. Then the following statements hold:

i) $p \in \mathcal{O} \cap \hat{\mathcal{S}}$ is a Lyapunov stable fixed point of (7) if and only if the periodic orbit \mathcal{O} of \mathcal{G} generated by p is Lyapunov stable.

ii) $p \in \mathcal{O} \cap \hat{\mathcal{S}}$ is an asymptotically stable fixed point of (7) if and only if the periodic orbit \mathcal{O} of \mathcal{G} generated by p is asymptotically stable.

Proof. *i)* To show necessity, let $\epsilon > 0$ and note that the set $\mathcal{S}_p = \{x \in \mathcal{D} : x = s(\tau_l(p), p) = p_l, l = 1, \dots, N\}$ contains N points, where $p \triangleq p_1$. Furthermore, let $p^+ = \lim_{\tau \rightarrow 0} s(\tau, p)$ and let $\hat{\epsilon} > 0$. It follows from joint continuity of solutions of \mathcal{G} that there exists $\hat{\delta} = \hat{\delta}(p, \hat{\epsilon})$ such that if $\|x'_0 - p^+\| + |t - t'| < \hat{\delta}$, then $\|s(t, x'_0) - s(t', p^+)\| < \hat{\epsilon}$, where $t \in (0, \tau_1(x'_0)]$ and $t' \in (0, \tau_1(p^+)]$. Next, as shown in the proof of Proposition 2.2, it follows that $\lim_{\lambda \rightarrow 0} \underline{\tau}_1(\lambda, p^+) = \lim_{\lambda \rightarrow 0} \bar{\tau}_1(\lambda, p^+) = \tau_1(p^+)$. Hence, choosing $\delta' = \delta'(p, \hat{\epsilon}) > 0$ such that $\delta' < \frac{\hat{\delta}}{2}$ and $\bar{\tau}_1(\delta', p^+) - \underline{\tau}_1(\delta', p^+) + \mu < \frac{\hat{\delta}}{2}$, where μ is a sufficiently small constant, it follows from the joint continuity property that, since $\|x'_0 - p^+\| + |t - \tau_1(p^+)| < \hat{\delta}$,

$$\|s(t, x'_0) - s(\tau_1(p^+), p^+)\| < \hat{\epsilon}, \quad x'_0 \in \mathcal{B}_{\delta'}(p^+), \quad t \in [\underline{\tau}_1(\delta', p^+) - \mu, \tau_1(x'_0)]. \quad (8)$$

Next, let $\hat{\eta} > 0$ be such that $\hat{\eta} < \tau_1(p^+) - \underline{\tau}_1(\delta', p^+) + \mu$. Now, it follows from the strong quasi-continuous dependence property *iv)*' that there exists $\delta'' = \delta''(p, \hat{\epsilon}) > 0$ such that

$$\|s(t, p^+) - s(t, x'_0)\| < \hat{\epsilon}, \quad x'_0 \in \mathcal{B}_{\delta''}(p^+), \quad t \in [0, \tau_1(p^+) - \hat{\eta}]. \quad (9)$$

Now, let $\tilde{\delta} = \min\{\delta', \delta''\}$ and note that $\underline{\tau}_1(\delta', p^+) \leq \underline{\tau}_1(\tilde{\delta}, p^+)$. Since $\hat{\eta}$ is such that $\underline{\tau}_1(\delta', p^+) - \mu \in [0, \tau_1(p^+) - \hat{\eta}]$, it follows from (8) and (9) that

$$\text{dist}(s(t, x'_0), \mathcal{O}) < \hat{\epsilon}, \quad x'_0 \in \mathcal{B}_{\tilde{\delta}}(p^+), \quad t \in [0, \tau_1(x'_0)]. \quad (10)$$

Using similar arguments as above, it can be shown that the resetting event is continuous with respect to the state; that is, for $\tilde{\delta} > 0$ there exists $\delta_1 = \delta_1(\tilde{\delta}) = \delta_1(\hat{\epsilon})$ such that $\|x^+ - p^+\| < \tilde{\delta}$ for all $x \in \mathcal{B}_{\delta_1}(p) \cap \mathcal{S}$, where $x^+ = \lim_{\tau \rightarrow 0} s(\tau, x)$. Hence, it follows that for $\hat{\epsilon} > 0$, there exists $\delta_1 = \delta_1(\hat{\epsilon})$ such that

$$\text{dist}(s(t, x'_0), \mathcal{O}) < \hat{\epsilon}, \quad x'_0 \in \mathcal{B}_{\delta_1}(p) \cap \mathcal{S}, \quad t \in [0, \tau_2(x'_0)]. \quad (11)$$

Similarly for every point in \mathcal{S}_p there exists a neighborhood such that an analogous condition to (11) holds. Specifically, for $\epsilon > 0$ and $p_N \in \mathcal{S}_p$ there exists $\delta_N = \delta_N(\epsilon) < \epsilon$ such that $\text{dist}(s(t, x'_0), \mathcal{O}) < \epsilon$ for all $x'_0 \in \mathcal{B}_{\delta_N}(p_N) \cap \mathcal{S}$, $t \in [0, \tau_2(x'_0)]$. Analogously, for $p_{N-1} \in \mathcal{S}_p$, there exists $\delta_{N-1} = \delta_{N-1}(\delta_N) = \delta_{N-1}(\epsilon)$ such that $\text{dist}(s(t, x'_0), \mathcal{O}) < \delta_N < \epsilon$ for all $x'_0 \in \mathcal{B}_{\delta_{N-1}}(p_{N-1}) \cap \mathcal{S}$, $t \in [0, \tau_2(x'_0)]$. Recursively repeating this procedure and using the semi-group property *iii)*, it follows that for $\epsilon > 0$ there exists $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$\text{dist}(s(t, x'_0), \mathcal{O}) < \epsilon, \quad x'_0 \in \mathcal{B}_{\delta_1}(p) \cap \mathcal{S}, \quad t \in [0, \tau_{N+1}(x'_0)]. \quad (12)$$

Next, it follows from Lyapunov stability of the fixed point $p \in \mathcal{S}_p$ of the discrete-time dynamical system (7) that, for $\delta_1 > 0$, there exists $\delta'_1 = \delta'_1(\delta_1) > 0$ such that $\|z(k+1) - p\| = \|P(z(k)) - p\| < \delta_1$ for all $z(0) \in \mathcal{B}_{\delta'_1}(p) \cap \mathcal{S}$. Hence, using (12) and the semi-group property *iii)*, it follows that

$$\text{dist}(s(t, x'_0), \mathcal{O}) < \epsilon, \quad x'_0 \in \mathcal{B}_{\delta'_1}(p) \cap \mathcal{S}, \quad t \geq 0. \quad (13)$$

Using similar arguments as above, for every $p_{\mathcal{O}} \in \overline{\mathcal{O}}$ there exists $\delta = \delta(\epsilon, p_{\mathcal{O}})$ such that

$$\text{dist}(s(t, x_0), \mathcal{O}) < \epsilon, \quad x_0 \in \mathcal{B}_{\delta}(p_{\mathcal{O}}), \quad t \in [0, \tau_m(p_{\mathcal{O}})], \quad (14)$$

where m is the number of resettings required for $s(t, x_0), t \geq 0$, to reach $\mathcal{B}_{\delta_1}(p) \cap \mathcal{S}$. Finally, it follows from (14), (13), and the semi-group property *iii*) that

$$\text{dist}(s(t, x_0), \mathcal{O}) < \epsilon, \quad x_0 \in \mathcal{B}_\delta(p_{\mathcal{O}}), \quad t \geq 0, \quad (15)$$

which, using Lemma 2.1, proves Lyapunov stability of the periodic orbit \mathcal{O} .

Next, we show sufficiency. Assume that \mathcal{O} is a Lyapunov stable periodic orbit. Furthermore, choose $\epsilon > 0$ and let $\hat{\epsilon} \in (0, \epsilon]$ be such that there does not exist a point of \mathcal{S}_p in $\mathcal{B}_{\hat{\epsilon}}(p)$ other than $p \in \mathcal{S}_p$. Note that $\hat{\epsilon} > 0$ exists since \mathcal{S}_p is a finite set. Now, using the fact that \mathcal{G} is left-continuous, it follows that for sufficiently small $\hat{\delta} > 0$ there exists $\tilde{\delta} = \tilde{\delta}(\hat{\delta})$ such that $\hat{\delta} \leq \tilde{\delta} < \hat{\epsilon}$ and

$$\text{dist}(x, \mathcal{O}) > \hat{\delta}, \quad x \in \mathcal{B}_{\tilde{\delta}}(p) \setminus \mathcal{B}_{\hat{\delta}}(p) \cap \mathcal{S}. \quad (16)$$

Here, we let $\tilde{\delta} > 0$ be the smallest value such that (16) holds. Note that in this case

$$\lim_{\hat{\delta} \rightarrow 0} \tilde{\delta}(\hat{\delta}) = 0. \quad (17)$$

Now, it follows from Assumption 2.1 and the joint continuity of solutions of \mathcal{G} that for $\hat{\epsilon} > 0$ there exists $\delta'(\hat{\epsilon}) > 0$ such that

$$\|s(\tau_{N+1}(x'_0), x'_0) - p\| < \hat{\epsilon}, \quad x'_0 \in \mathcal{B}_{\delta'}(p) \cap \hat{\mathcal{S}}. \quad (18)$$

Hence, using (17), we can choose $\hat{\delta} = \hat{\delta}(\hat{\epsilon}) > 0$ such that $\tilde{\delta}(\hat{\delta}) \leq \delta'(\hat{\epsilon})$. Next, it follows from the Lyapunov stability of \mathcal{O} that for $\hat{\delta} = \hat{\delta}(\hat{\epsilon}) > 0$ there exists $\delta = \delta(\hat{\delta}) = \delta(\hat{\epsilon}) > 0$ such that if $x_0 \equiv z(0) \in \mathcal{B}_{\delta}(p) \cap \hat{\mathcal{S}}$, then $\text{dist}(s(t, x_0), \mathcal{O}) < \hat{\delta}, t \geq 0$. Now, using (16) and (18), it follows that

$$\begin{aligned} \|z(k+1) - p\| &= \|P(z(k)) - p\| = \|s(\tau_{N+1}(z(k)), z(k)) - p\| < \tilde{\delta} < \hat{\epsilon} \leq \epsilon, \\ z(0) &\in \mathcal{B}_{\delta}(p) \cap \hat{\mathcal{S}}, \quad k \in \mathcal{N}, \end{aligned} \quad (19)$$

where $z(k), k \in \mathcal{N}$, satisfies (7). Thus, (19) establishes that $p \in \mathcal{S}_p$ is a Lyapunov stable fixed point of (7).

ii) To show necessity, assume that $p \in \mathcal{O} \cap \hat{\mathcal{S}}$ is an asymptotically stable fixed point of (7). Hence, it follows from *i*) that the periodic orbit \mathcal{O} is Lyapunov stable and there exists $\delta' > 0$

such that if $z(0) \equiv x'_0 \in \mathcal{B}_{\delta'}(p) \cap \hat{\mathcal{S}}$, then $z(k+1) = P(z(k)) = s(\tau_{(N+1) \cdot (k+1)}(x'_0), x'_0) \rightarrow p$ as $k \rightarrow \infty$. Now, Definition 2.5 implies that a periodic orbit \mathcal{O} of \mathcal{G} is asymptotically stable if it is Lyapunov stable and there exists $\delta > 0$ such that if $\text{dist}(x'_0, \mathcal{O}) < \delta$, then for every $\epsilon > 0$ there exists $T = T(\epsilon, x'_0) > 0$ such that $\text{dist}(s(t, x'_0), \mathcal{O}) < \epsilon$ for all $t > T$. Next, using similar arguments as in *i*), for any $\epsilon > 0$ there exists $\hat{\delta} = \hat{\delta}(\epsilon) > 0$ such that

$$\text{dist}(s(t, x'_0), \mathcal{O}) < \epsilon, \quad x'_0 \in \mathcal{B}_{\hat{\delta}}(p) \cap \hat{\mathcal{S}}, \quad t \in [0, \tau_{N+1}(x'_0)]. \quad (20)$$

Now, it follows from the asymptotic stability of p that for every $x'_0 \in \mathcal{B}_{\delta'}(p) \cap \hat{\mathcal{S}}$ there exists $K = K(\hat{\delta}, x'_0) = K(\epsilon, x'_0) \in \mathcal{N}$ such that

$$\|s(\tau_{(N+1) \cdot k}(x'_0), x'_0) - p\| < \hat{\delta}, \quad k > K. \quad (21)$$

Choose $l > K$ and let $T = T(\epsilon, x'_0) = \tau_{(N+1) \cdot l}(x'_0)$. Then, it follows from (20) and (21) that for a given $\epsilon > 0$ there exists $T = T(\epsilon, x'_0) > 0$ such that

$$\text{dist}(s(t, x'_0), \mathcal{O}) < \epsilon, \quad x'_0 \in \mathcal{B}_{\delta'}(p) \cap \hat{\mathcal{S}}, \quad t > T, \quad (22)$$

and hence $\text{dist}(s(t, x'_0), \mathcal{O}) \rightarrow 0$ as $t \rightarrow \infty$ for all $x'_0 \in \mathcal{B}_{\delta'}(p) \cap \hat{\mathcal{S}}$.

Finally, using similar arguments as in *i*) it can be shown that for every $p_{\mathcal{O}} \in \overline{\mathcal{O}}$ there exists $\delta = \delta(p_{\mathcal{O}}) > 0$ such that $\|s(\tau_m(x_0), x_0) - p\| < \delta'$ for all $x_0 \in \mathcal{B}_{\delta}(p_{\mathcal{O}})$, where m is the number of resettings required for $s(t, x_0), t \geq 0$, to reach $\mathcal{B}_{\delta'}(p) \cap \hat{\mathcal{S}}$. This argument along with (22), the semi-group property *iii*), and *ii*) of Lemma 2.1 implies asymptotic stability of the periodic orbit \mathcal{O} .

Finally, we show sufficiency. Assume that \mathcal{O} is an asymptotically stable periodic orbit of \mathcal{G} . Hence, $p \in \hat{\mathcal{S}}$ is a Lyapunov stable fixed point of (7) and there exists $\delta > 0$ such that if $x_0 \in \mathcal{B}_p(\delta)$, then for every sequence $\{t_k\}_{k=0}^{\infty}$ such that $t_k \rightarrow \infty$ as $k \rightarrow \infty$,

$$\text{dist}(s(t_k, x_0), \mathcal{O}) \rightarrow 0, \quad k \rightarrow \infty. \quad (23)$$

Next, choose $\hat{\delta} \in (0, \delta]$ such that there are no points of \mathcal{S}_p in $\mathcal{B}_{\hat{\delta}}(p)$ other than $p \in \hat{\mathcal{S}}$. Once again, $\hat{\delta} > 0$ exists since \mathcal{S}_p is a finite set. Since $p \in \hat{\mathcal{S}}$ is a Lyapunov stable fixed point of (7) it follows that for $\hat{\delta} > 0$ there exists $\tilde{\delta} = \tilde{\delta}(\hat{\delta}) > 0$ such that if $z(0) \equiv x_0 \in \mathcal{B}_{\tilde{\delta}}(p) \cap \hat{\mathcal{S}}$, then $z(k+1) = P(z(k)) = s(\tau_{(N+1) \cdot (k+1)}(x_0), x_0) \in \mathcal{B}_{\hat{\delta}}(p) \cap \hat{\mathcal{S}}, k \in \mathcal{N}$. Next, choose a sequence

$\{t_k\}_{k=0}^\infty = \{\tau_{(N+1)\cdot k}(x_0)\}_{k=0}^\infty$, $x_0 \in \mathcal{B}_{\hat{\delta}}(p) \cap \hat{\mathcal{S}}$, and note that $\tau_{(N+1)\cdot k}(x_0) \rightarrow \infty$ as $k \rightarrow \infty$. Hence, it follows from (23) that

$$\begin{aligned} \text{dist}(z(k+1), \mathcal{O}) = \text{dist}(P(z(k)), \mathcal{O}) = \text{dist}(s(\tau_{(N+1)\cdot(k+1)}(x_0), x_0), \mathcal{O}) \rightarrow 0, \quad k \rightarrow \infty, \\ x_0 \in \mathcal{B}_{\hat{\delta}}(p) \cap \hat{\mathcal{S}}. \end{aligned} \quad (24)$$

Since $p \in \mathcal{S}_p$ is the only point of \mathcal{O} in $\mathcal{B}_{\hat{\delta}}(p) \cap \hat{\mathcal{S}}$, (24) implies that $\text{dist}(z(k+1), p) \rightarrow 0$ as $k \rightarrow \infty$ for all $z(0) \equiv x_0 \in \mathcal{B}_{\hat{\delta}}(p) \cap \hat{\mathcal{S}}$, which establishes asymptotic stability of the fixed point $p \in \mathcal{S}_p$ of (7). \square

3. Specialization to Impulsive Dynamical Systems

In this section we specialize Poincaré's theorem for strong left-continuous dynamical systems to state-dependent impulsive dynamical systems [5–10]. A state-dependent impulsive dynamical system \mathcal{G} has the form

$$\dot{x}(t) = f_c(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{S}, \quad (25)$$

$$\Delta x(t) = f_d(x(t)), \quad x(t) \in \mathcal{S}, \quad (26)$$

where $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, \mathcal{D} is an open set, $f_c : \mathcal{D} \rightarrow \mathbb{R}^n$, $f_d : \mathcal{S} \rightarrow \mathbb{R}^n$ is continuous, and $\mathcal{S} \subset \mathcal{D}$ is the *resetting set*. We refer to the differential equation (25) as the *continuous-time dynamics*, and we refer to the difference equation (26) as the *resetting law*. We assume that the continuous-time dynamics $f_c(\cdot)$ are such that the solution to (25) is jointly continuous in t and x_0 between resetting events. A sufficient condition ensuring this is Lipschitz continuity of $f_c(\cdot)$. Alternatively, uniqueness of solutions in forward time along with the continuity of $f_c(\cdot)$ ensure that solutions to (25) between resetting events are continuous functions of the initial conditions $x_0 \in \mathcal{D}$ even when $f_c(\cdot)$ is not Lipschitz continuous on \mathcal{D} (see [15], Theorem 4.3, p. 59). More generally, $f_c(\cdot)$ need not be continuous. In particular, if $f_c(\cdot)$ is discontinuous but bounded and $x(\cdot)$ is the unique solution to (25) between resetting events in the sense of Filippov [16], then continuous dependence of solutions with respect to the initial conditions holds [16]. Finally, note that since the resetting set \mathcal{S} is a subset of the state space \mathcal{D} and is independent of time, state-dependent impulsive dynamical systems are time-invariant.

As in Section 2, for a particular trajectory $x(t)$ we denote the resetting times of (25), (26) by $\tau_k(x_0)$; that is, the k^{th} instant of time at which $x(t)$ intersects \mathcal{S} . Thus the trajectory of the system (25), (26) from the initial condition $x(0) = x_0 \in \mathcal{D}$ is given by $\psi(t, x_0)$ for $0 < t \leq \tau_1(x_0)$, where $\psi(t, x_0)$ denotes the solution to the continuous-time dynamics (25). If and when the trajectory reaches a state $x_1 \triangleq x(\tau_1(x_0))$ satisfying $x_1 \in \mathcal{S}$, then the state is instantaneously transferred to $x_1^+ \triangleq x_1 + f_d(x_1)$ according to the resetting law (26). The solution $x(t), \tau_1(x_0) < t \leq \tau_2(x_0)$, is then given by $\psi(t - \tau_1(x_0), x_1^+)$, and so on for all $x_0 \in \mathcal{D}$. Note that the solution $x(t)$ of (25), (26) is left-continuous; that is, it is continuous everywhere except at the resetting time $\tau_k(x_0)$, and

$$x_k \triangleq x(\tau_k(x_0)) = \lim_{\epsilon \rightarrow 0^+} x(\tau_k(x_0) - \epsilon), \quad (27)$$

$$x_k^+ \triangleq x(\tau_k(x_0)) + f_d(x(\tau_k(x_0))), \quad (28)$$

for $k = 1, 2, \dots$.

We make the following additional assumptions:

A.1. If $x(t) \in \bar{\mathcal{S}} \setminus \mathcal{S}$, where $\bar{\mathcal{S}}$ denotes the closure of the set \mathcal{S} , then there exists $\epsilon > 0$ such that, for all $0 < \delta < \epsilon$, $\psi(\delta, x(t)) \notin \mathcal{S}$.

A.2. If $x \in \mathcal{S}$, then $x + f_d(x) \notin \mathcal{S}$.

Assumption A.1 ensures that if a trajectory reaches the closure of \mathcal{S} at a point that does not belong to \mathcal{S} , then the trajectory must be directed away from \mathcal{S} ; that is, a trajectory cannot enter \mathcal{S} through a point that belongs to the closure of \mathcal{S} but not to \mathcal{S} . Furthermore, A.2 ensures that when a trajectory intersects the resetting set \mathcal{S} , it instantaneously exits \mathcal{S} . Finally, we note that if $x_0 \in \mathcal{S}$, then the system initially resets to $x_0^+ = x_0 + f_d(x_0) \notin \mathcal{S}$ which serves as the initial condition for the continuous-time dynamics (25).

Remark 3.1. It follows from A.2 that resetting removes $x(\tau_k(x_0)) = x_k$ from the resetting set \mathcal{S} . Thus, immediately after resetting occurs, the continuous-time dynamics (25), and not the resetting law (26), becomes the active element of the impulsive dynamical system.

Remark 3.2. It follows from A.1 and A.2 that no trajectory starting outside of \mathcal{S} can intersect the interior of \mathcal{S} . Specifically, it follows from A.1 that a trajectory can only reach \mathcal{S} through a point belonging to both \mathcal{S} and its boundary. And from A.2, it follows that

if a trajectory reaches a point in \mathcal{S} that is on the boundary of \mathcal{S} , then the trajectory is instantaneously removed from \mathcal{S} . Since a continuous trajectory starting outside of \mathcal{S} and intersecting the interior of \mathcal{S} must first intersect the boundary of \mathcal{S} , it follows that no trajectory can reach the interior of \mathcal{S} .

Remark 3.3. Note that it follows from the definition of $\tau_k(\cdot)$ that $\tau_1(x) > 0, x \notin \mathcal{S}$, and $\tau_1(x) = 0, x \in \mathcal{S}$. Furthermore, since for every $x \in \mathcal{S}, x + f_d(x) \notin \mathcal{S}$, it follows that $\tau_2(x) = \tau_1(x) + \tau_1(x + f_d(x)) > 0$.

Remark 3.4. It follows from A.1, A.2, and Remark 3.2 that the resetting times $\tau_k(x_0)$ are well defined and distinct. See [2, 9] for further details of this fact.

It is important to note that since *not* every bounded solution of an impulsive dynamical system over a forward time interval can be extended to infinity due to Zeno solutions; that is, solutions exhibiting infinitely many resettings in a finite time, we assume that $f_c(\cdot)$ and $f_d(\cdot)$ are such that existence and uniqueness of solutions for (25), (26) are satisfied in forward time. For details see [2, 5, 8]. In this paper we assume that $f_c(\cdot)$ and $f_d(\cdot)$ are such that $\tau_k(x_0) \rightarrow \infty$ as $k \rightarrow \infty$ for all $x_0 \in \mathcal{D}$. In light of the above, note that the solution of (25), (26) with initial condition $x_0 \in \mathcal{D}$ denoted by $s(t, x_0), t \geq 0$, is *i*) left-continuous; that is, $\lim_{\tau \rightarrow t^-} s(\tau, x_0) = s(t, x_0)$ for all $x_0 \in \mathcal{D}$ and $t \in (0, \infty)$; *ii*) consistent; that is, $s(0, x_0) = x_0$, for all $x_0 \in \mathcal{D}$; and *iii*) satisfies the semi-group property; that is, $s(\tau, s(t, x_0)) = s(t + \tau, x_0)$ for all $x_0 \in \mathcal{D}$ and $t, \tau \in [0, \infty)$. To see this note that $s(0, x_0) = x_0$ for all $x_0 \in \mathcal{D}$ and

$$s(t, x_0) = \begin{cases} \psi(t, x_0), & 0 \leq t \leq \tau_1(x_0), \\ \psi(t - \tau_k(x_0), s(\tau_k(x_0), x_0) + f_d(s(\tau_k(x_0), x_0))), & \tau_k(x_0) < t \leq \tau_{k+1}(x_0), \\ \psi(t - \tau(x_0), s(\tau(x_0), x_0)), & t \geq \tau(x_0), \end{cases} \quad (29)$$

where $\tau(x_0) \triangleq \sup_{k \geq 0} \tau_k(x_0)$, which implies that $s(\cdot, x_0)$ is left-continuous. Furthermore, uniqueness of solutions implies that $s(t, x_0)$ satisfies the semi-group property $s(\tau, s(t, x_0)) = s(t + \tau, x_0)$ for all $x_0 \in \mathcal{D}$ and $t, \tau \in [0, \infty)$.

Next, we present two key assumptions on the structure of the resetting set \mathcal{S} . Specifically, we assume that the resetting set \mathcal{S} is such that the following assumptions hold:

A.3. There exists a continuously differentiable function $\mathcal{X} : \mathcal{D} \rightarrow \mathbb{R}$ such that the resetting set $\mathcal{S} = \{x \in \mathcal{D} : \mathcal{X}(x) = 0\}$; moreover, $\mathcal{X}'(x) \neq 0, x \in \mathcal{S}$.

A.4. $\frac{\partial \mathcal{X}(x)}{\partial x} f_c(x) \neq 0, x \in \mathcal{S}$.

Remark 3.5. It follows from A.3 that the resetting set \mathcal{S} is an embedded submanifold [17] while A.4 assures that the solution of \mathcal{G} is not tangent to the resetting set \mathcal{S} .

The following lemma proven in [12] states that under the Assumptions A.3 and A.4, the first resetting time $\tau_1(\cdot)$ is continuous at $x_0 \in \mathcal{D}$.

Lemma 3.1 [12]. Consider the nonlinear state-dependent impulsive dynamical system \mathcal{G} given by (25), (26). Assume that A.3 and A.4 hold. Then $\tau_1(\cdot)$ is continuous at $x_0 \in \mathcal{D}$, where $0 < \tau_1(x_0) < \infty$.

The following proposition shows that under Assumptions A.3 and A.4, the resetting times $\tau_k(\cdot)$ are continuous at $x_0 \in \mathcal{D}$ for all $k \in \mathcal{N}$.

Proposition 3.1. Consider the nonlinear state-dependent impulsive dynamical system \mathcal{G} given by (25), (26). Assume that A.3 and A.4 hold. Then $\tau_k(\cdot)$ is continuous at $x_0 \in \mathcal{D}$, where $0 < \tau_k(x_0) < \infty$, for all $k \in \mathcal{N}$.

Proof. First, it follows from Lemma 3.1 that $\tau_1(\cdot)$ is continuous at $x_0 \in \mathcal{D}$, where $0 < \tau_1(x_0) < \infty$. Since $f_c(\cdot)$ is such that the solutions to (25) are jointly continuous in t and x_0 , it follows that $\psi(\cdot, \cdot)$ is continuous in both its arguments. Furthermore, note that $\psi(\tau_1(x), x) = s(\tau_1(x), x), x \in \mathcal{D}$. Next, it follows from the definition of $\tau_k(x)$ that for every $x \in \mathcal{D}$ and $k \in \{1, 2, \dots, \}$,

$$\tau_k(x) = \tau_{k-j}(x) + \tau_j[s(\tau_{k-j}(x), x) + f_d(s(\tau_{k-j}(x), x))], \quad j = 1, \dots, k, \quad (30)$$

where $\tau_0(x) \triangleq 0$. Hence, since $f_d(\cdot)$ is continuous, it follows from (30) that $\tau_2(x) = \tau_1(x) + \tau_1[s(\tau_1(x), x) + f_d(s(\tau_1(x), x))]$ is also continuous on \mathcal{D} . By recursively repeating this procedure for $k = 3, 4, \dots$, it follows that $\tau_k(x)$ is a continuous function on \mathcal{D} for all $k \in \mathcal{N}$. \square

Since $f_c(\cdot)$ and $f_d(\cdot)$ are such that the Axioms *i) – iii)* hold for the state-dependent impulsive dynamical system \mathcal{G} , and \mathcal{G} is jointly continuous between resetting events, then,

with Assumptions A.3 and A.4 satisfied, it follows from Propositions 2.2 and 3.1 that the state-dependent impulsive dynamical system \mathcal{G} is a strong left-continuous dynamical system. Hence, the following corollary to Theorem 2.1 is immediate.

Corollary 3.1. Consider the impulsive dynamical system \mathcal{G} given by (25), (26) with the Poincaré return map defined by (6). Assume that A.3 and A.4 hold, and the point $p \in \hat{\mathcal{S}}$ generates the periodic orbit $\mathcal{O} \triangleq \{x \in \mathcal{D} : x = s(t, p), 0 \leq t \leq T\}$, where $s(t, p)$, $t \geq 0$, is the periodic solution with the period $T = \tau_{N+1}(p)$ such that $s(\tau_{N+1}(p), p) = p$. Then the following statements hold:

- i) $p \in \mathcal{O} \cap \hat{\mathcal{S}}$ is a Lyapunov stable fixed point of (7) if and only if the periodic orbit \mathcal{O} of \mathcal{G} generated by p is Lyapunov stable.
- ii) $p \in \mathcal{O} \cap \hat{\mathcal{S}}$ is an asymptotically stable fixed point of (7) if and only if the periodic orbit \mathcal{O} of \mathcal{G} generated by p is asymptotically stable.

Corollary 3.1 gives necessary and sufficient conditions for Lyapunov and asymptotic stability of a periodic orbit of the state-dependent impulsive dynamical system \mathcal{G} based on the stability properties of a fixed point of the n -dimensional discrete-time dynamical system involving the Poincaré map (6). Next, as is the case of the classical Poincaré theorem, we present a specialization of Corollary 3.1 that allows us to analyze the stability of periodic orbits by replacing the n th-order impulsive dynamical system by an $(n - 1)$ th-order discrete-time system. To present this result assume, without loss of generality, that $\frac{\partial \mathcal{X}(x)}{\partial x_n} \neq 0, x \in \mathcal{S}$, where $x = [x_1, \dots, x_n]^T$. Then it follows from the implicit function theorem [18] that $x_n = g(x_1, \dots, x_{n-1})$, where $g(\cdot)$ is a continuously differentiable function at $x_r \triangleq [x_1, \dots, x_{n-1}]^T$ such that $[x_r^T, g(x_r)]^T \in \mathcal{S}$. Note that in this case $P : \hat{\mathcal{S}} \rightarrow \mathcal{S}$ in (7) is given by $P(x) \triangleq [P_1(x), \dots, P_n(x)]^T$, where

$$P_n(x_r, g(x_r)) = g(P_1(x_r, g(x_r)), \dots, P_{n-1}(x_r, g(x_r))). \quad (31)$$

Hence, we can reduce the n -dimensional discrete-time system (7) to the $(n - 1)$ -dimensional discrete-time system given by

$$z_r(k + 1) = \mathcal{P}_r(z_r(k)), \quad k \in \mathcal{N}, \quad (32)$$

where $z_r \in \mathbb{R}^{n-1}$, $[z_r^T(\cdot), g(z_r(\cdot))]^T \in \mathcal{S}$, and

$$\mathcal{P}_r(x_r) \triangleq \begin{bmatrix} P_1(x_r, g(x_r)) \\ \vdots \\ P_{n-1}(x_r, g(x_r)) \end{bmatrix}. \quad (33)$$

Note that it follows from (31) and (33) that $p \triangleq [p_r^T, g(p_r)]^T \in \hat{\mathcal{S}}$ is a fixed point of (7) if and only if p_r is a fixed point of (32).

Corollary 3.2. Consider the impulsive dynamical system \mathcal{G} given by (25), (26) with the Poincaré return map defined by (6). Assume that A.3 and A.4 hold, $\frac{\partial \mathcal{X}(x)}{\partial x_n} \neq 0$, $x \in \mathcal{S}$, and the point $p \in \hat{\mathcal{S}}$ generates the periodic orbit $\mathcal{O} \triangleq \{x \in \mathcal{D} : x = s(t, p), 0 \leq t \leq T\}$, where $s(t, p)$, $t \geq 0$, is the periodic solution with the period $T = \tau_{N+1}(p)$ such that $s(\tau_{N+1}(p), p) = p$. Then the following statements hold:

- i)* For $p = [p_r^T, g(p_r)]^T \in \mathcal{O} \cap \hat{\mathcal{S}}$, p_r is a Lyapunov stable fixed point of (32) if and only if the periodic orbit \mathcal{O} is Lyapunov stable.
- ii)* For $p = [p_r^T, g(p_r)]^T \in \mathcal{O} \cap \hat{\mathcal{S}}$, p_r is an asymptotically stable fixed point of (32) if and only if the periodic orbit \mathcal{O} is asymptotically stable.

Proof. *i)* To show necessity, assume that p_r is a Lyapunov stable fixed point of (32) and let $\epsilon > 0$. Then it follows from the continuity of $g(\cdot)$ that there exists $\delta' = \delta'(\epsilon) > 0$ such that

$$\|g(x_r) - g(p_r)\| < \frac{\epsilon}{2}, \quad x_r \in \mathcal{B}_{\delta'}(p_r). \quad (34)$$

Choosing $\delta' < \frac{\epsilon}{2}$, it follows from the Lyapunov stability of p_r that for $\delta' > 0$ there exists $\delta = \delta(\epsilon) < \delta'$ such that

$$\|z_r(k+1) - p_r\| = \|\mathcal{P}_r(z_r(k)) - p_r\| < \delta' < \frac{\epsilon}{2}, \quad z_r(0) \in \mathcal{B}_\delta(p_r), \quad (35)$$

where $z(k) = [z_r(k)^T, g(z_r(k))]^T \in \mathcal{S}$, $k \in \mathcal{N}$, satisfies (7). Hence, if $z(0) \in \mathcal{B}_\delta(p) \cap \hat{\mathcal{S}}$; that is,

$$\begin{aligned} \|z(0) - p\| &= \|[z_r(0)^T, g(z_r(0))]^T - [p_r^T, g(p_r)]^T\| \leq \|[z_r(0)^T, 0]^T - [p_r^T, 0]^T\| \\ &\quad + \|[0, \dots, 0, g(z_r(0))]^T - [0, \dots, 0, g(p_r)]^T\| < \delta, \end{aligned} \quad (36)$$

then it follows from (36) that $z_r(0) \in \mathcal{B}_\delta(p_r)$. Hence, using (31), (34), and (35), it follows that

$$\begin{aligned} \|z(k+1) - p\| &= \|P(z(k)) - p\| = \|[\mathcal{P}_r^T(z_r(k)), P_n(z_r(k), g(z_r(k)))]^T - [p_r^T, g(p_r)]^T\| \\ &\leq \|[\mathcal{P}_r^T(z_r(k)), 0]^T - [p_r^T, 0]^T\| \\ &\quad + \|[0, \dots, 0, P_n(z_r(k), g(z_r(k)))]^T - [0, \dots, 0, g(p_r)]^T\| < \delta' + \frac{\epsilon}{2} < \epsilon. \end{aligned} \quad (37)$$

Thus, for a given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $z(0) \in \mathcal{B}_\delta(p) \cap \hat{\mathcal{S}}$, then $\|z(k+1) - p\| < \epsilon$, which establishes Lyapunov stability of p for the discrete-time system (7). Now, Lyapunov stability of the periodic orbit \mathcal{O} follows as a direct consequence of Theorem 2.1.

Next, to show sufficiency, assume that the periodic orbit \mathcal{O} is Lyapunov stable. In this case, it follows from Theorem 2.1 that $p = [p_r^T, g(p_r)]^T \in \mathcal{O} \cap \hat{\mathcal{S}}$ is a Lyapunov stable fixed point of (7). Hence, for every $\epsilon > 0$ there exists $\delta' = \delta'(\epsilon) > 0$ such that

$$\|z(k+1) - p\| = \|P(z(k)) - p\| < \epsilon, \quad k \in \mathcal{N}, \quad z(0) \in \mathcal{B}_{\delta'}(p) \cap \hat{\mathcal{S}}. \quad (38)$$

Now, it follows from the continuity of $g(\cdot)$ that for $\delta' > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$\|g(x_r) - g(p_r)\| < \frac{\delta'}{2}, \quad x_r \in \mathcal{B}_\delta(p_r). \quad (39)$$

Choosing $\delta < \frac{\delta'}{2}$ and $z_r(0) \in \mathcal{B}_\delta(p_r)$, it follows from (39) that

$$\|z(0) - p\| = \|[z_r(0)^T, g(z_r(0))]^T - [p_r^T, g(p_r)]^T\| < \frac{\delta'}{2} + \frac{\delta'}{2} = \delta'; \quad (40)$$

that is, $z(0) \in \mathcal{B}_{\delta'}(p) \cap \hat{\mathcal{S}}$. Hence, it follows from (38), (40), and (37) that for a given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|z_r(k+1) - p_r\| = \|\mathcal{P}_r(z_r(k)) - p_r\| < \epsilon$ for all $z_r(0) \in \mathcal{B}_\delta(p_r)$, which establishes Lyapunov stability of p_r for (32).

ii) To show necessity, assume that p_r is an asymptotically stable fixed point of (32). Now, it follows from *i)* that $p = [p_r^T, g(p_r)]^T \in \mathcal{O} \cap \hat{\mathcal{S}}$ is a Lyapunov stable fixed point of (7) and there exists $\delta > 0$ such that

$$\|z_r(k+1) - p_r\| = \|\mathcal{P}_r(z_r(k)) - p_r\| \rightarrow 0, \quad k \rightarrow \infty, \quad z_r(0) \in \mathcal{B}_\delta(p_r). \quad (41)$$

If $z(0) \in \mathcal{B}_\delta(p) \cap \hat{\mathcal{S}}$, then, as shown in *i)*, $z_r(0) \in \mathcal{B}_\delta(p_r)$. Hence, it follows from (41), the continuity of $g(\cdot)$, and representation given in (37) and (31) that $\|z(k+1) - p\| \rightarrow 0$ as

$k \rightarrow \infty$ for all $z(0) \in \mathcal{B}_\delta(p) \cap \hat{\mathcal{S}}$. This establishes asymptotic stability of p for (7). Now, asymptotic stability of the periodic orbit \mathcal{O} of \mathcal{G} follows as a direct consequence of Theorem 2.1.

Finally, to show sufficiency, assume that the periodic orbit \mathcal{O} is asymptotically stable. In this case, it follows from Theorem 2.1 that $p = [p_r^T, g(p_r)]^T \in \mathcal{O} \cap \hat{\mathcal{S}}$ is an asymptotically stable fixed point of (7) and, by *i*), p_r is a Lyapunov stable fixed point of (32). Hence, there exists $\delta' > 0$ such that

$$\|z(k+1) - p\| = \|P(z(k)) - p\| = \|[\mathcal{P}_r^T(z_r(k)), P_n(z_r(k), g(z_r(k)))]^T - [p_r^T, g(p_r)]^T\| \rightarrow 0, \\ k \rightarrow \infty, \quad z(0) \in \mathcal{B}_{\delta'}(p) \cap \hat{\mathcal{S}}. \quad (42)$$

Using similar arguments as in *i*), we can find $\delta > 0$ such that if $z_r(0) \in \mathcal{B}_\delta(p_r)$, then $z(0) \in \mathcal{B}_{\delta'}(p) \cap \hat{\mathcal{S}}$. Thus it follows from (42) that

$$\|z_r(k+1) - p_r\| = \|\mathcal{P}_r(z_r(k)) - p_r\| \rightarrow 0, \quad k \rightarrow \infty, \quad z_r(0) \in \mathcal{B}_\delta(p_r), \quad (43)$$

which establishes asymptotic stability of p_r for (32). \square

4. Conclusion

A generalized Poincaré theorem was developed for left-continuous dynamical systems. Specifically, necessary and sufficient conditions for stability of left-continuous, hybrid, and impulsive dynamical systems with periodic solutions were derived based on stability properties of a fixed point of a discrete-time dynamical system constructed from a Poincaré return map.

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