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Stability and dissipativity theory for discrete-time non-negative and compartmental dynamical systems

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Non-negative and compartmental dynamical systems are derived from mass and energy balance considerations that involve dynamic states whose values are non-negative. These models are widespread in engineering, biomedicine and ecology. In this paper we develop several results on stability, dissipativity and stability of feedback interconnections of discrete-time linear and non-linear non-negative dynamical systems. Specifically, using *linear* Lyapunov functions we first develop necessary and sufficient conditions for Lyapunov stability and asymptotic stability for non-negative systems. In addition, using *linear* and non-linear storage functions with *linear* supply rates we develop new notions of dissipativity theory for non-negative dynamical systems. Finally, these results are used to develop general stability criteria for feedback interconnections of non-negative dynamical systems.

1. Introduction

Non-negative dynamical systems (Luenberger 1979, Nieuwenhuis 1982, Ohta *et al*. 1984, Berman *et al*. 1989, Farina and Rinaldi 2000, Kaszkurewicz and Bhaya 2000, Haddad *et al*. 2001, Kaczorek 2002) are essential in capturing the phenomenological features of a wide range of dynamical systems. These systems are derived from mass and energy balance considerations that involve dynamic states whose values are non-negative. Hence, it follows from physical considerations that the state trajectories of such systems remain in the non-negative orthant of the state space for non-negative initial conditions. A subclass of non-negative dynamical systems are compartmental systems (Mohler 1974, Maeda *et al*. 1977, 1978, Sandberg 1978, Funderlic and Mankin 1981, Anderson 1983, Godfrey 1983, Jacquez 1985, Bernstein and Hyland 1993, Jacquez and Simon 1993). These systems are comprised of homogeneous coupled macroscopic subsystems or compartments which exchange variable quantities of material via intercompartmental flow laws. The range of applications of non-negative and compartmental system is widespread in science and engineering. Their usage include biomedical systems (Jacquez 1985, Jacquez and Simon 1993), demographic, epidemic and ecological systems (Mulholland and Keener 1974, Jacquez *et al*. 1998), economic systems (Berman and Plemmons 1979), telecommunication systems (Foster and Garzia 1989), transportation systems, power systems and large-scale systems (Siljak 1978, 1983).

Even though numerous results on continuous-time compartmental systems and, to a lesser extent, non-negative systems have been developed in the literature (see Mohler 1974, Maeda *et al*. 1977, 1978, Sandberg 1978, Funderlic and Mankin 1981, Nieuwenhuis 1982, Anderson 1983, Godfrey 1983, Ohta *et al*. 1984, Jacquez 1985, Berman *et al*. 1989, Bernstein and Hyland 1993, Jacquez and Simon 1993, Farina and Rinaldi 2000, Kaszkurewicz and Bhaya 2000, Haddad *et al*. 2001, Kaczorek 2002 and references therein), the development of discrete-time non-negative and compartmental systems theory has received far less attention. In this paper we develop several basic mathematical results on stability, dissipativity and stability of feedback interconnections of discrete-time linear and non-linear non-negative dynamical systems. Specifically, using *linear* Lyapunov functions, we first develop necessary and sufficient conditions for Lyapunov stability and asymptotic stability for linear non-negative dynamical systems. It is important to note that linear Lyapunov functions for non-negative dynamical systems is not new to this paper and have been considered in Luenberger (1979). However, the consideration of a linear Lyapunov function in the present formulation leads to a *new* Lyapunov-like equation for examining the stability of linear non-negative systems. This Lyapunov-like equation is analysed using non-negative matrix theory (Berman and Plemmons 1979, Horn and Johnson 1995). The motivation for using a linear Lyapunov function follows from the fact that the state of a non-negative dynamical system is non-negative and hence a linear Lyapunov function is a valid Lyapunov function candidate. This considerably simplifies the stability analysis of non-negative dynamical systems.

It is well known that linear, continuous-time Lyapunov stable compartmental systems are semistable (Bernstein and Hyland 1993, Jacquez and Simon 1993, Haddad *et al*. 2001); that is, system trajectories converge

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to Lyapunov stable equilibrium points. This is not the case however for linear, discrete-time compartmental systems as well as non-linear compartmental systems. In fact, non-linear compartmental systems can exhibit limit cycles, bifurcations and even chaos (Jacquez and Simon 1993). Hence, it is of interest to determine necessary and sufficient conditions under which masses/concentrations for non-linear compartmental systems converge. Even though sufficient conditions do exist in the literature guaranteeing the absence of limit cycles for non-linear, continuous-time compartmental systems (Maeda *et al.* 1978, Hirsch 1982, Chellaboina *et al.* 2002), similar conditions for discrete-time compartmental systems are not developed. In this paper we additionally develop necessary and sufficient conditions for identifying discrete-time non-negative and compartmental systems that only admit monotonic solutions. As a result we provide new sufficient conditions for the absence of limit cycles in non-linear, discrete-time compartmental systems.

Next, using *linear* and *non-linear* storage functions with *linear* supply rates we develop *new* notions of classical dissipativity theory (Willems 1972 a) for discrete-time non-negative dynamical systems. The overall approach provides a new interpretation of a mass balance for non-negative systems with linear supply rates and linear and non-linear storage functions. Specifically, we show that dissipativity of a non-negative dynamical system involving a linear storage function and a linear supply rate implies that the system mass transport is equal to the supplied system mass minus the expelled system mass. In the special case where the linear supply rate is taken to be the excess input mass flux over the output mass flux, the system dissipativity notion collapses to a *non-accumulativity* system constraint wherein the system mass transport is always less than or equal to the difference between the system flux input and system flux output. Furthermore, we show that all compartmental systems with measured outputs corresponding to material outflows are non-accumulative. In addition, we develop *new* Kalman–Yakubovich–Popov equations for non-negative systems for characterizing dissipativeness with linear and non-linear storage functions and linear supply rates. Finally, using the concepts of dissipativity with linear supply rates we develop feedback interconnection stability results for discrete-time linear and non-linear non-negative dynamical systems. General stability criteria are given for Lyapunov and asymptotic stability of feedback non-negative dynamical systems. These results can be viewed as a generalization of the positivity and the small gain theorems (Hill and Moylan 1977) to non-negative systems with linear supply rates involving net input–output system mass flux.

The contents of the paper are as follows. In § 2 we establish definitions, notation and review some basic results on non-negative matrices. In § 3 we present Lyapunov and asymptotic stability results for discrete-time linear non-negative dynamical systems using a linear Lyapunov function and a new Lyapunov-like equation. We then turn our attention to stability and convergence results for discrete-time non-linear non-negative dynamical systems in § 4. In § 5, we present several results for discrete-time non-negative dynamical systems with non-negative inputs and non-negative outputs. Furthermore, we extend the notation of dissipativity theory to discrete-time non-negative dynamical systems with linear storage functions and linear supply rates. In addition, we develop new Kalman–Yakubovich–Popov equations in terms of system storage functions and linear supply rates for characterizing dissipativeness for discrete-time non-negative systems. In § 6, we specialize the results of § 5 to linear discrete-time non-negative systems. In § 7, we use the results of §§ 5 and 6 to state and prove feedback interconnection stability results for dissipative discrete-time non-linear non-negative dynamical systems. Furthermore, we develop absolute stability criteria for discrete-time non-negative dynamical systems with non-negative memoryless feedback non-linearities. Section 8 considers an example that demonstrates the utility of some of the mathematical results developed in the paper. Finally, we draw conclusions in § 9. We stress that although many of the system theory results and stability results on *linear* non-negative dynamical systems in this paper are well known (Luenberger 1979, Farina and Rinaldi 2000, Kaszkurewicz and Bhaya 2000), they are restated here in a concise and unified format that supports the development of dissipativity theory for non-negative systems in later sections.

Notation

$\mathcal{N}, \mathbb{R}, \mathbb{R}^{m\times n}$	non-negative integers, real numbers, $m \times n$ real matrices
x_i	ith entry of vector $x \in \mathbb{R}^n$
e_i	vector with unity in ith position and zeros elsewhere
e	$[1, 1, \ldots, 1]^{\mathrm{T}}$
$A_{(i,j)}$	(i, j)th entry of matrix $A \in \mathbb{R}^{m\times n}$
$A \geq\geq 0$ ($A \gg 0$)	$A_{(i,j)} \geq 0$ ($A_{(i,j)} > 0$) for all i and j
$A \geq\geq B$ ($A \gg B$)	$A - B \geq\geq 0$ ($A - B >> 0$) where A and B are matrices with identical dimensions

$A \geq 0$ ($A > 0$)	non-negative (resp., positive) definite matrix; that is, symmetric matrix with non-negative (resp., positive) eigenvalues
$A \geq B$ ($A > B$)	$A - B \geq 0\,(A - B > 0)$ where A and B are symmetric matrices with identical dimensions
$\mathbb{R}^n_+, \overline{\mathbb{R}}^n_+,$	$\{x \in \mathbb{R}^n\colon\ x >> 0\}, \{x \in \mathbb{R}^n\colon\ x \geq\geq 0\}$
$\text{spec}(A)$	spectrum of matrix $A \in \mathbb{R}^{n\times n}$
$\rho(A)$	spectral radius of A; that is, $\max\{\lvert\lambda\rvert\colon\ \lambda \in \text{spec}(A)\}$
rank A	rank of matrix $A \in \mathbb{R}^{m\times n}$
$\text{ind}(A)$	$\min\{k \in \mathcal{N}\colon\ \text{rank}\ A^k = \text{rank}\ A^{k+1}\}$
$\det(A)$	determinant of matrix $A \in \mathbb{R}^{n\times n}$
$\text{row}_i(A), \text{col}_i(A)$	ith row of A, ith column of A
A^{T}	transpose of A
$A^{\#}$	group generalized inverse of A where $\text{ind}(A) \leq 1$
$\mathcal{R}(A), \mathcal{N}(A)$	range and null subspaces of $A \in \mathbb{R}^{m\times n}$
$A \otimes B$	Kronecker product of A and B
$\lVert\cdot\rVert$	vector or matrix norm
$\mathcal{B}_r(x_0)$	$\{x \in \mathbb{R}^n\colon\ \lVert x - x_0\rVert < r\}$
$\overline{\mathcal{S}}$	closure of the set $\mathcal{S}$

2. Mathematical preliminaries

In this section we introduce several definitions and some key results concerning non-negative matrices (Berman *et al.* 1978, Meyer and Stadelmaier 1978, Berman and Plemmons 1979, Horn and Johnson 1995) that are necessary for developing the main results of this paper.

Definition 1: Let $A \in \mathbb{R}^{m\times n}$. Then A is *non-negative*† (resp., *positive*) if $A_{(i,j)} \geq 0$ (resp., $A_{(i,j)} > 0$) for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Definition 2: A real function $u\colon\mathcal{N} \to \mathbb{R}^m$ is a *non-negative* (resp., *positive*) *function* if $u(k) \geq\geq 0$ (resp., $u(k) >> 0$), $k \in \mathcal{N}$.

Definition 3 (Berman and Plemmons 1979)**:** Let $A \in \mathbb{R}^{n\times n}$. A is a *Z-matrix* if $A_{(i,j)} \leq 0$, $i, j = 1, \ldots, n$, $i \neq j$. A is an *M-matrix* (resp., a *non-singular M-matrix*) if A is a *Z-matrix* and all the principal minors of A are non-negative (resp., positive). A is *essentially non-negative* if $-A$ is a *Z-matrix*; that is, $A_{(i,j)} \geq 0$, $i, j = 1, \ldots, n$, $i \neq j$.

† In this paper it is important to distinguish between a square non-negative (resp., positive) matrix and a non-negetive-definite (resp., positive-definite) matrix.

The following lemma is needed for developing several stability results in later sections.

Lemma 1: *Let $A \in \mathbb{R}^{n\times n}$ be non-negative. Then the following statements are equivalent:*

(i) *$I - A$ is an M-matrix.*
(ii) *$\rho(A) \leq 1$.*

Furthermore, if $I - A \in \mathbb{R}^{n\times n}$ is a non-singular Z-matrix, then the following statements are equivalent:

(iii) *$I - A$ is a non-singular M-matrix.*
(iv) $\det(I - A) \neq 0$ and $(I - A)^{-1} \geq\geq 0$.
(v) *For each $y \in \mathbb{R}^n$, $y \geq\geq 0$, there exists a unique $x \in \mathbb{R}^n$, $x \geq\geq 0$, such that $(I - A)x = y$.*
(vi) *There exists $x \in \mathbb{R}^n$, $x \geq\geq 0$, such that $x >> Ax$.*
(vii) *There exists $x \in \mathbb{R}^n$, $x >> 0$, such that $x >> Ax$.*

Proof: To show (i) implies (ii) let $I - A$ be an M-matrix and note that it follows from (iii) of Lemma 2.1 of Haddad *et al.* (2001) that $\text{Re}\,\lambda \geq 0$, $\lambda \in \text{spec}\,(I - A)$ or, equivalently, $\text{Re}\ \lambda \leq 1$, $\lambda \in \text{spec}\,(A)$. Since A is non-negative it follows from the Perron–Frobenius theorem (Berman and Plemmons 1979, Horn and Johnson 1995) that there exists $\lambda \in \text{spec}\,(A)$ such that $\rho(A) = \lambda$ which implies that $\rho(A) = \text{Re}\,\lambda \leq 1$. Conversely, to show (ii) implies (i) let $\rho(A) \leq 1$. Now it follows from (ii) of Lemma 2.1 of Haddad *et al.* (2001) with $\alpha = 1$, $B = A$, and A replaced by $I - A$ that $I - A$ is an M-matrix. The equivalence of statements (iii)–(vii) follows from Theorem A.2 of Siljak (1978) with A replaced by $I - A$. □

3. Stability theory for discrete-time linear non-negative dynamical systems

In this section we provide necessary and sufficient conditions for stability of discrete-time non-negative dynamical systems; that is, dynamical systems whose solutions remain in the non-negative orthant for non-negative initial conditions. Specifically, we consider linear dynamical systems of the form

$$x(k+1) = Ax(k), \qquad x(0) = x_0, \qquad k \in \mathcal{N} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $k \in \mathcal{N}$, and $A \in \mathbb{R}^{n\times n}$. Since the solution to (1) is given by $x(k) = A^k x_0$ it follows that $x(k) \geq\geq 0$, $k \in \mathcal{N}$, if and only if A is non-negative. Henceforth, we assume that A is non-negative. The following definition introduces several types of stability notions corresponding to the equilibrium solution

$x(k) \equiv x_e$ of (1), where $x_e \in \mathcal{N}(A - I)$, and whose solutions remain in the non-negative orthant $\overline{\mathbb{R}}^n_+$.

Definition 4: The equilibrium solution $x(k) \equiv x_e$ of the non-negative dynamical system (1) is *Lyapunov stable* if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_e) \cap \overline{\mathbb{R}}^n_+$, then $x(k) \in \mathcal{B}_\varepsilon(x_e) \cap \overline{\mathbb{R}}^n_+$, $k \in \mathcal{N}$. The equilibrium solution $x(k) \equiv x_e$ of the non-negative dynamical system (1) is *semistable* if it is Lyapunov stable and there exists $\delta > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_e) \cap \overline{\mathbb{R}}^n_+$, then $\lim_{k\to\infty} x(k)$ exists and converges to a Lyapunov stable equilibrium point. The equilibrium solution $x(k) \equiv x_e$ of the non-negative dynamical system (1) is *asymptotically stable* if it is Lyapunov stable and there exists $\delta > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_e) \cap \overline{\mathbb{R}}^n_+$, then $\lim_{k\to\infty} x(k) = x_e$. Finally, the equilibrium solution $x(k) \equiv x_e$ of the non-negative dynamical system (1) is *globally asymptotically stable* if the previous statement holds for all $x_0 \in \overline{\mathbb{R}}^n_+$.

Remark 1: Note that (1) is Lyapunov stable if and only if there exists $\alpha > 0$ such that $\|A^k\| \le \alpha$, $k \in \mathcal{N}$, (1) is semistable if and only if $\lim_{k\to\infty} A^k$ exists, and (1) is asymptotically stable if and only if $\lim_{k\to\infty} A^k = 0$. Also note that if A is asymptotically stable, then $\mathcal{N}(A - I) = \{0\}$.

The following theorem gives necessary and sufficient conditions for Lyapunov stability and asymptotic stability for discrete-time linear non-negative dynamical systems using *linear* Lyapunov functions. Even though *some* of the results presented below follow from non-negative matrix theory (Berman and Plemmons 1979), here we provide a proof based on standard Lyapunov theory and invariant set arguments using a linear Lyapunov function construction.

Theorem 1: *Consider the discrete-time linear dynamical system* (1) *where* $A \in \mathbb{R}^{n\times n}$ *is non-negative. Then the following statements hold:*

(i) *If there exists vectors* $p, r \in \mathbb{R}^n$ *such that* $p >> 0$ *and* $r \ge\ge 0$ *satisfy*

$$p = A^{\mathrm{T}} p + r \tag{2}$$

then A *is Lyapunov stable.*

(ii) *If* A *is Lyapunov stable, then there exists vectors* $p, r \in \mathbb{R}^n$ *such that* $p \ge\ge 0$, $p \ne 0$, *and* $r \ge\ge 0$ *satisfy* (2).

(iii) *If there exist vectors* $p, r \in \mathbb{R}^n$ *such that* $p \ge\ge 0$ *and* $r \ge\ge 0$ *satisfy* (2) *and* (A, r^{T}) *is observable, then* $p >> 0$ *and* (1) *is asymptotically stable.*

Furthermore, the following statements are equivalent:

(iv) A *is asymptotically stable.*

(v) *There exist vectors* $p, r \in \mathbb{R}^n$ *such that* $p >> 0$ *and* $r >> 0$ *satisfy* (2).

(vi) *There exist vectors* $p, r \in \mathbb{R}^n$ *such that* $p \ge\ge 0$ *and* $r >> 0$ *satisfy* (2).

(vii) *For every* $r \in \mathbb{R}^n$ *such that* $r >> 0$, *there exists* $p \in \mathbb{R}^n$ *such that* $p >> 0$ *satisfies* (2).

Proof:

(i) Consider the linear Lyapunov function candidate $V(x) = p^{\mathrm{T}} x$. Note that $V(0) = 0$ and $V(x) > 0$, $x \in \overline{\mathbb{R}}^n_+$, $x \ne 0$. Now, computing the Lyapunov difference yields

$$\Delta V(x) \triangleq V(Ax) - V(x) = p^{\mathrm{T}} Ax - p^{\mathrm{T}} x = -r^{\mathrm{T}} x \le 0, \quad x \in \overline{\mathbb{R}}^n_+$$

establishing Lyapunov stability.

(ii) If A is Lyapunov stable it follows from (i) of Lemma 1 that $(I - A)^{\mathrm{T}}$ is an M-matrix. Since $A^{\mathrm{T}} \ge\ge 0$, it follows from the Perron–Frobenius theorem (Berman and Plemmons 1979) that $\rho(A) \in \mathrm{spec}(A)$ and hence there exists $p \ge\ge 0$, $p \ne 0$, such that $A^{\mathrm{T}} p = \rho(A)p$. Thus, $A^{\mathrm{T}} p - p = (\rho(A) - 1)p \le\le 0$ which proves that there exist $p \ge\ge 0$, $p \ne 0$, and $r \ge\ge 0$ such that (2) holds.

(iii) Assume there exist $p \ge\ge 0$ and $r \ge\ge 0$ such that (2) holds and suppose (A, r^{T}) is observable. Now, consider the function $V(x) = p^{\mathrm{T}} x$, $x \in \overline{\mathbb{R}}^n_+$, and note that since $V(x) \ge 0$ and $\Delta V(x) = p^{\mathrm{T}} Ax - p^{\mathrm{T}} x = -r^{\mathrm{T}} x \le 0$, $x \in \overline{\mathbb{R}}^n_+$, it follows that if $x(0) \in \mathcal{P} \triangleq \{x \in \overline{\mathbb{R}}^n_+ : p^{\mathrm{T}} x = 0\}$, then $V(x(k)) = 0$, $k \in \mathcal{N}$, which implies that $0 \ge \Delta V(x(k)) = p^{\mathrm{T}} Ax(k) - p^{\mathrm{T}} x(k) = p^{\mathrm{T}} Ax(k) \ge 0$, $k \in \mathcal{N}$. Specifically, $\Delta V(x(0)) = p^{\mathrm{T}} Ax(0) = 0$. Hence, if $\hat{x} \in \mathcal{P}$ then $\Delta V(\hat{x}) = p^{\mathrm{T}} A\hat{x} = -r^{\mathrm{T}} \hat{x} = 0$. Thus, if $\hat{x} \in \mathcal{P}$ then $A\hat{x} \in \mathcal{P}$ and $\hat{x} \in \mathcal{Q} \triangleq \{x \in \overline{\mathbb{R}}^n_+ : r^{\mathrm{T}} x = 0\}$. Now, since $A\hat{x} \in \mathcal{P}$ it follows that $A^2\hat{x} \in \mathcal{P}$ and $A\hat{x} \in \mathcal{Q}$. Repeating these arguments yields $A^k \hat{x} \in \mathcal{Q}$, $k = 0, 1, \ldots, n-1$, or, equivalently, $r^{\mathrm{T}} A^k \hat{x} = 0$, $k = 0, 1, \ldots, n-1$. Now, since (A, r^{T}) is observable it follows that $\hat{x} = 0$ and $\mathcal{P} = \{0\}$ which implies that $p >> 0$. Asymptotic stability of (1) now follows as a direct consequence of LaSalle's invariant set theorem (LaSalle 1976) with $V(x) = p^{\mathrm{T}} x$ and using the fact that (A, r^{T}) is observable.

To show the equivalence between (iv)–(vii) first suppose there exists $p \ge\ge 0$ and $r >> 0$ such that (2) holds. Now, there exists sufficiently small $\varepsilon > 0$ such that $A^{\mathrm{T}}(p + \varepsilon e) << p + \varepsilon e$ and $p + \varepsilon e >> 0$ which proves that (vi) implies (v). Since (v) implies (vi) it trivially follows that (v) and (vi) are equivalent. Now, suppose

(v) holds; that is, there exists $p >> 0$ and $r >> 0$ such that (2) holds and consider the Lyapunov function candidate $V(x) = p^{\mathrm{T}} x$, $x \in \overline{\mathbb{R}}_{+}^{n}$. Computing the Lyapunov difference yields $\Delta V(x) = p^{\mathrm{T}} Ax - p^{\mathrm{T}} x = -r^{\mathrm{T}} x < 0$, $x \in \overline{\mathbb{R}}_{+}^{n}$, $x \neq 0$, and hence it follows that (1) is asymptotically stable. Thus, (v) implies (iv). Next, suppose (1) is asymptotically stable. Hence, $(I - A)^{-\mathrm{T}} \geq\geq 0$ and thus for every $r \in \mathbb{R}_{+}^{n}$, $p \triangleq (I - A)^{-\mathrm{T}} r \geq\geq 0$ satisfies (2) which proves that (iv) implies (vi).

Finally, suppose (1) is asymptotically stable. Now, as in the proof given above, for every $r \in \mathbb{R}_{+}^{n}$, there exists $p \in \overline{\mathbb{R}}_{+}^{n}$ such that (2) holds. Next, suppose, *ad absurdum*, there exists $x \in \overline{\mathbb{R}}_{+}^{n}$, $x \neq 0$, such that $x^{\mathrm{T}} p = 0$; that is, there is at least one $i \in \{1, 2, \ldots, n\}$, such that $p_i = 0$. Hence, $x^{\mathrm{T}}(I - A)^{-\mathrm{T}} r = 0$. However, since $(I - A)^{-\mathrm{T}} \geq\geq 0$ it follows that $(I - A)^{-1} x \geq\geq 0$ and, since $r >> 0$, it follows that $(I - A)^{-1} x = 0$ which implies that $x = 0$ yielding a contradiction. Hence, for every $r \in \mathbb{R}_{+}^{n}$, there exists $p \in \mathbb{R}_{+}^{n}$ such that (2) holds which proves (iv) implies (vii). Since (vii) implies (v) trivially, the equivalence of (iv)–(vii) is established. □

Next, we give necessary and sufficient conditions for semistability of a non-negative matrix.

Theorem 2: *Let $A \in \mathbb{R}^{n \times n}$ be non-negative. A is semistable if and only if $|\lambda| < 1$ or $\lambda = 1$ and $\lambda = 1$ is semisimple, where $\lambda \in \operatorname{spec}(A)$. Furthermore, if A is semistable, then* $\lim_{k \to \infty} A^k = I - (A - I)(A - I)^{\#} \geq\geq 0$.

Proof: If $|\lambda| < 1$ or $\lambda = 1$ and $\lambda = 1$ is semisimple, then A is Lyapunov stable. Now, it follows from the Jordan decomposition that there exists an invertible matrix $S \in \mathbb{C}^{n \times n}$ such that

$$A = S \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} S^{-1}$$

where $J \in \mathbb{C}^{r \times r}$, $r = \operatorname{rank} A$, and $\rho(J) < 1$. Hence, it follows that

$$\begin{aligned} \lim_{k \to \infty} A^k &= \lim_{k \to \infty} S \begin{bmatrix} J^k & 0 \\ 0 & I \end{bmatrix} S^{-1} \\ &= S \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S^{-1} \\ &= I - S \begin{bmatrix} J - I & 0 \\ 0 & 0 \end{bmatrix} S^{-1} S \begin{bmatrix} (J - I)^{-1} & 0 \\ 0 & 0 \end{bmatrix} S^{-1} \\ &= I - (A - I)(A - I)^{\#} \end{aligned}$$

which implies that $\lim_{k \to \infty} A^k$ exists. Hence, A is semistable. Furthermore, since $A^k \geq\geq 0$, $k \in \mathcal{N}$, it follows that $I - (A - I)(A - I)^{\#} \geq\geq 0$.

Conversely, suppose A is semistable; that is, $\lim_{k \to \infty} A^k$ exists. Since A is semistable it follows that A is Lyapunov stable; that is, if $\lambda \in \operatorname{spec}(A)$, then either $|\lambda| < 1$, or $|\lambda| = 1$ and λ is semisimple. Now, it follows from the Jordan decomposition that there exists an invertible matrix $S \in \mathbb{C}^{n \times n}$ such that

$$A = S \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} S^{-1}$$

where $J_1 \in \mathbb{C}^{r \times r}$ such that $\rho(J_1) < 1$ and $J_2 \in \mathbb{C}^{(n-r) \times (n-r)}$ is diagonal such that $|\lambda| = 1$, $\lambda \in \operatorname{spec}(J_2)$. Hence,

$$A^k = S \begin{bmatrix} J_1^k & 0 \\ 0 & J_2^k \end{bmatrix} S^{-1}$$

Now, since for $\lambda \in \mathbb{C}$, $|\lambda| = 1$, $\lim_{k \to \infty} \lambda^k$ exists if and only if $\lambda = 1$, it follows that $\lim_{k \to \infty} A^k$ exists if and only if $J_2 = I$, which proves the result. □

It is important to note that unlike the case of continuous-time linear non-negative systems (Haddad *et al.* 2001), Lyapunov stability of discrete-time linear non-negative systems does *not* necessarily imply semistability of a discrete-time linear non-negative system. To see this, let $n = 2$ and

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so that A is non-negative and discrete-time Lyapunov stable. In this case, $\lim_{k \to \infty} A^k$ does not exist which shows that A is not semistable. However, if the discrete-time linear non-negative system is derived from the discretization of a continuous-time linear non-negative semistable system, then it can be shown that the discrete-time system is semistable. To see this, consider the continuous-time linear non-negative system

$$\dot{x}(t) = A_{\mathrm{c}} x(t), \qquad x(0) = x_0, \qquad t \geq 0 \tag{3}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $A_{\mathrm{c}} \in \mathbb{R}^{n \times n}$ is essentially non-negative and semistable (Haddad *et al.* 2001), so that the discretization of (3) (with sampling rate $h = 1$) is given by

$$x(k+1) = A_{\mathrm{d}} x(k), \qquad x(0) = x_0, \qquad k \in \mathcal{N} \tag{4}$$

where $A_{\mathrm{d}} = \mathrm{e}^{A_{\mathrm{c}}}$. In this case, since A_{c} is (continuous-time) semistable, it follows from the real Jordan decomposition that there exists an invertible matrix $S \in \mathbb{R}^{n \times n}$ such that

$$A_{\mathrm{c}} = S \begin{bmatrix} J_{\mathrm{c}} & 0 \\ 0 & 0 \end{bmatrix} S^{-1}$$

where J_{c} is Hurwitz, so that

$$A_{\mathrm{d}} = \mathrm{e}^{A_{\mathrm{c}}} = S \begin{bmatrix} e^{J_{\mathrm{c}}} & 0 \\ 0 & I \end{bmatrix} S^{-1}$$

Since $\rho(\mathrm{e}^{J_{\mathrm{c}}}) < 1$, it follows that if $\lambda \in \operatorname{spec}(A_{\mathrm{d}})$, then either $|\lambda| < 1$, or $\lambda = 1$ and λ is semisimple. Now, it follows from Theorem 2 that A_d is (discrete-time) semistable.

Next, using Theorem 1, we show that every asymptotically stable discrete-time linear non-negative system is equivalent, modulo a similarity transformation, to a compartmental system. We note that this result is well known (see, e.g. Farina and Rinaldi (2000, p. 147)). However, here we give a *new* proof of this result based on the Lyapunov-like equation (2).

Proposition 1: *Let $A \in \mathbb{R}^{n\times n}$ be non-negative and asymptotically stable. Then there exists a diagonal invertible matrix $S \in \mathbb{R}^{n\times n}$ such that $\hat{A}_{(i,j)} \geq 0$ and $\sum_{k=1}^{n} \hat{A}_{(k,j)} \leq 1$, $i, j = 1, \ldots, n$, where $\hat{A} = SAS^{-1}$.*

Proof: It follows from (iv) and (v) of Theorem 1 that there exists $p \in \mathbb{R}_+^n$ such that $A^{\mathrm{T}}p - p << 0$. Now, define $S \triangleq \mathrm{diag}[p_1, \ldots, p_n]$, where p_i is the ith component of p. Next, since $\hat{A}_{(i,j)} = p_i A_{(i,j)} p_j^{-1}$, $i, j = 1, \ldots, n$, it follows that $\hat{A}_{(i,j)} \geq 0$, $i, j = 1, \ldots, n$. Furthermore, since $S\hat{A}^{\mathrm{T}}e - Se = A^{\mathrm{T}}Se - p = A^{\mathrm{T}}p - p << 0$ which, since $S >> 0$ and diagonal, implies that $\hat{A}^{\mathrm{T}}e << e$. Hence, $\sum_{k=1}^{n} \hat{A}_{(k,j)} \leq 1$, $i, j = 1, \ldots, n$. □

The next results give necessary and sufficient conditions for asymptotic stability of a discrete-time linear non-negative dynamical system using *quadratic* Lyapunov functions. The first result appears in Kaszkurewicz and Bhaya (2000) and Farina and Rinaldi (2000) and is stated here for completeness, while the second result is the discrete-time analogue of Theorem A.14 in Siljak (1978).

Theorem 3: *Consider the dynamical system given by* (1) *where $A \in \mathbb{R}^{n\times n}$ is non-negative. Then* (1) *is asymptotically stable if and only if there exists a positive diagonal matrix $P \in \mathbb{R}^{n\times n}$ and an $n \times n$ positive-definite matrix R such that*

$$P = A^{\mathrm{T}}PA + R \tag{5}$$

Proof: See Kaszkurewicz and Bhaya (2000, p. 66). □

Theorem 4: *Consider the dynamical system given by* (1) *where $A \in \mathbb{R}^{n\times n}$ is non-negative. Then* (1) *is asymptotically stable if and only if for every positive, positive-definite $n \times n$ matrix R, there exists a positive, positive-definite $n \times n$ matrix P such that* (5) *holds.*

Proof: Sufficiency follows from standard discrete-time Lyapunov theory with Lyapunov function $V(x) = x^{\mathrm{T}}Px$. Conversely, assume A is non-negative and asymptotically stable and let R be a positive, positive-definite matrix (i.e. $R > 0$ and $R >> 0$). Since A is asymptotically stable it follows that there exists a unique $n \times n$ positive-definite matrix P such that (5) is satisfied and $\mathrm{vec}\, P = (A \otimes A)^{\mathrm{T}} \mathrm{vec}\, P + \mathrm{vec}\, R$, where $\otimes$ denotes Kronecker product and $\mathrm{vec}(\cdot)$ denotes the column stacking operator. Next, since A is non-negative and asymptotically stable it follows that $A \otimes A$ is non-negative and asymptotically stable. Now, since $R >> 0$, it follows that $\mathrm{vec}\, R >> 0$ and hence (vii) of Theorem 1 implies that $\mathrm{vec}\, P >> 0$ which establishes that $P >> 0$. □

Finally, we show that discrete-time linear compartmental dynamical systems (Mohler 1974, Maeda *et al.* 1977, 1978, Sandberg 1978, Funderlic and Mankin 1981, Anderson 1983, Godfrey 1983, Jacquez 1985, Bernstein and Hyland 1993, Jacquez and Simon 1993) are a special case of discrete-time non-negative dynamical systems. To see this, let $x_i(k)$, $k \in \mathcal{N}$, $i = 1, \ldots, n$, denote the mass (and hence a non-negative quantity) of the ith subsystem of the compartmental system shown in figure 1, let $a_{ii} \geq 0$ denote the loss coefficient of the ith subsystem averaged over the discretization interval h, let $w_i(k) \geq 0$, $k \in \mathcal{N}$, $i = 1, \ldots, n$, denote the mass inflow supplied to the ith subsystem and let $\phi_{ij}(k)$, $k \in \mathcal{N}$, $i \neq j$, $i, j = 1, \ldots, n$, denote the net mass flow from the jth subsystem to the ith subsystem given by $\phi_{ij}(k) = a_{ij}x_j(k) - a_{ji}x_i(k)$, $k \in \mathcal{N}$, where the transfer coefficient $a_{ij} \geq 0$, $i \neq j$, $i, j = 1, \ldots, n$, is averaged over the discretization interval h. Hence, a mass balance for the whole compartmental system with time

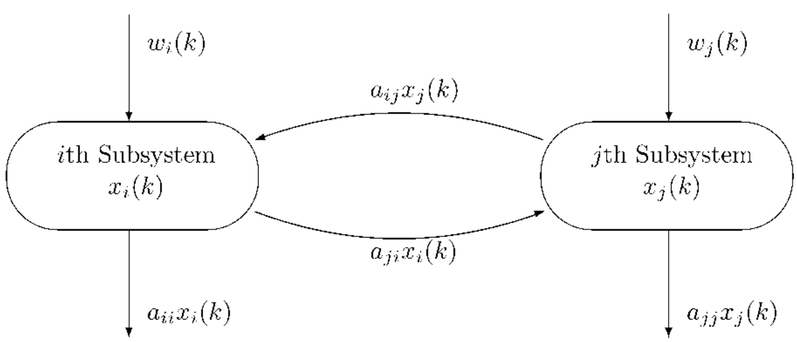

Figure 1. Discrete-time linear compartmental interconnected subsystem model.

step $h=1$ yields

$$\Delta x_i(k) \triangleq x_i(k+1) - x_i(k) = -a_{ii}x_i(k) + \sum_{j=1,\, i\neq j}^{n} \phi_{ij}(k) + w_i(k), \qquad k \in \mathcal{N}, \quad i = 1,\ldots,n \tag{6}$$

or, equivalently

$$x(k+1) = Ax(k) + w(k), \qquad x(0) = x_0, \quad k \in \mathcal{N} \tag{7}$$

where $x(k) = [x_1(k),\ldots,x_n(k)]^{\mathrm{T}}, w(k) = [w_1(k),\ldots,w_n(k)]^{\mathrm{T}}$, and for $i, j = 1,\ldots,n$

$$A_{(i,j)} = \begin{cases} 1 - \sum_{l=1}^{n} a_{li}, & i = j \\ a_{ij}, & i \neq j \end{cases} \tag{8}$$

Note that since at any given instant of time mass can only be transported, stored or discharged but not created and the maximum amount of mass that can be transported and/or discharged cannot exceed the mass in a compartment, it follows that $1 \geq \sum_{l=1}^{n} a_{li}$. Thus A is a non-negative matrix and hence the compartmental model given by (6) is a non-negative dynamical system. Furthermore, note that $A^{\mathrm{T}}e = [-a_{11}, -a_{22},\ldots,-a_{nn}]^{\mathrm{T}} + e$ and hence, with $p = e$ and $r = (I-A)^{\mathrm{T}}e \geq\geq 0$, it follows that (2) is satisfied, which implies that the compartmental model given by (6) ($w(k) \equiv 0$) is Lyapunov stable. In this case, $V(x) = e^{\mathrm{T}}x = \sum_{i=1}^{n} x_i$ denoting the total mass of the system serves as a Lyapunov function for the undisturbed ($w(k) \equiv 0$) system (6) with $\Delta V(x) = V(Ax) - V(x) = \sum_{i=1}^{n}(1-a_{ii})x_i - \sum_{i=1}^{n} x_i = -\sum_{i=1}^{n} a_{ii} \times x_i \leq 0$, $x \in \overline{\mathbb{R}}_+^n$. The compartmental system (6) with no inflows; that is, $w_i(k) \equiv 0$, $i = 1,\ldots,n$, is said to be *inflow-closed* (Jacquez 1985). Alternatively, if (6) possesses no losses (outflows) it is said to be *outflow-closed* (Jacquez 1985). A compartmental system is said to be *closed* if it is inflow-closed and outflow-closed. Note that for a closed system $\Delta V(x) = 0$, $x \in \overline{\mathbb{R}}_+^n$, which shows that the total mass inside a closed system is conserved. Alternatively, it follows that (6) can be equivalently written as

$$\Delta x(k) = [J_n(x(k)) - D(x(k))]\left(\frac{\partial V}{\partial x}(x(k))\right)^{\mathrm{T}} + w(k), \qquad x(0) = x_0, \quad k \in \mathcal{N} \tag{9}$$

where $J_n(x)$ is a skew-symmetric matrix function with $J_{n(i,i)}(x) = 0$ and $J_{n(i,j)}(x) = a_{ij}x_j - a_{ji}x_i$, $i \neq j$, and $D(x) = \mathrm{diag}[a_{11}x_1, a_{22}x_2,\ldots,a_{nn}x_n] \geq\geq 0$, $x \in \overline{\mathbb{R}}_+^n$. Hence, a discrete-time linear compartmental system is a discrete-time port-controlled Hamiltonian system with a Hamiltonian $\mathcal{H}(x) = V(x) = e^{\mathrm{T}}x$ representing the total mass in the system, $D(x)$ representing the outflow dissipation and $w(k)$, $k \in \mathcal{N}$, representing the supplied mass to the system. This observation shows that discrete-time compartmental systems are conservative systems. This will be further elaborated on in § 5.

4. Stability theory for discrete-time non-linear non-negative dynamical systems

In this section we consider discrete-time non-linear dynamical systems of the form

$$x(k+1) = f(x(k)), \qquad x(0) = x_0, \quad k \in \mathcal{N} \tag{10}$$

where $x(k) \in \mathcal{D}$, $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ with $0 \in \mathcal{D}$, and $f : \mathcal{D} \to \mathbb{R}^n$. Recall that the point $x_{\mathrm{e}} \in \mathcal{D}$ is an *equilibrium point* of (10) if $f(x_{\mathrm{e}}) = x_{\mathrm{e}}$. Furthermore, a subset $\mathcal{D}_{\mathrm{c}} \subseteq \mathcal{D}$ is an *invariant set* with respect to (10) if $\mathcal{D}_{\mathrm{c}}$ contains the orbits of all its points.

Definition 5: Let $f = [f_1,\ldots,f_n]^{\mathrm{T}} : \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ that contains $\overline{\mathbb{R}}_+^n$. Then f is *non-negative* if $f_i(x) \geq 0$, for all $i = 1,\ldots,n$, and $x \in \overline{\mathbb{R}}_+^n$.

Note that if $f(x) = Ax$, where $A \in \mathbb{R}^{n\times n}$, then f is non-negative if and only if A is non-negative. The following proposition shows that $\overline{\mathbb{R}}_+^n$ is an invariant set for (10) if and only if f is non-negative.

Proposition 2: *Suppose $\overline{\mathbb{R}}_+^n \subset \mathcal{D}$. Then $\overline{\mathbb{R}}_+^n$ is an invariant set with respect to* (10) *if and only if $f : \mathcal{D} \to \mathbb{R}^n$ is non-negative.*

Proof: Suppose $f : \mathcal{D} \to \mathbb{R}^n$ is non-negative and let $x(0) \in \overline{\mathbb{R}}_+^n$. Then, for every $i \in \{1,\ldots,n\}$ it follows that $x_i(k+1) = f_i(x(k)) \geq 0$. Thus, $x(k) \in \overline{\mathbb{R}}_+^n$, $k \in \mathcal{N}$. Conversely, suppose $x(k) \in \overline{\mathbb{R}}_+^n$, $k \in \mathcal{N}$, for all $x(0) \in \overline{\mathbb{R}}_+^n$ and assume, *ad absurdum*, that there exists $i \in \{1,\ldots,n\}$ and $x_0 \in \overline{\mathbb{R}}_+^n$ such that $f_i(x_0) < 0$. In this case, with $x(0) = x_0$, $x_i(1) = f_i(x(0)) = f_i(x_0) < 0$, which is a contradiction. □

The following result shows that if a non-linear system is non-negative, then its linearization is also non-negative.

Lemma 2: *Consider the non-linear dynamical system* (10) *where $f(0) = 0$ and $f : \mathcal{D} \to \mathbb{R}^n$ is non-negative and continuously differentiable in $\overline{\mathbb{R}}_+^n$. Then, $A \triangleq (\partial f/\partial x)(x)|_{x=0}$ is non-negative.*

Proof: Since $f : \mathcal{D} \to \mathbb{R}^n$ is non-negative it follows that $f_i(x) \geq 0$, $x \in \overline{\mathbb{R}}_+^n$. Now, note that for all $i, j \in \{1,\ldots,n\}$

$$A_{(i,j)} = \frac{\partial f_i}{\partial x_j}(x)\bigg|_{x=0} = \lim_{h\to 0^+} \frac{f_i(0,\ldots,h,\ldots,0) - f_i(0)}{h} = \lim_{h\to 0^+} \frac{f_i(0,\ldots,h,\ldots,0)}{h}$$

where h in $f_i(0,\dots,h,\dots,0)$ is on the jth location. Hence, since $f_i(\cdot)$ is non-negative, $A_{(i,j)} \geq 0$, which proves non-negativity of A. $\square$

Next, we present a key result on stability of a linearized discrete-time non-linear non-negative dynamical system. First, however, note that Definition 4 also holds for discrete-time non-linear non-negative dynamical systems. In this case, standard discrete-time Lyapunov stability theorems and invariant set theorems for non-linear systems (Vidyasagar 1993) can be used directly with the required sufficient conditions verified on $\overline{\mathbb{R}}^n_+$. Furthermore, the definition of a domain of attraction can be extended to discrete-time non-linear non-negative dynamical systems by restricting the domain to the non-negative orthant $\overline{\mathbb{R}}^n_+$. For details see Smith (1995).

Theorem 5: *Let $x(k) \equiv x_e$ be an equilibrium point for the non-linear dynamical system* (10). *Furthermore, let $f: \mathcal{D} \to \mathbb{R}^n$ be non-negative and let $A = (\partial f/\partial x)(x)|_{x=x_e}$. Then the following statements hold:*

(i) *If $|\lambda| < 1$, where $\lambda \in \operatorname{spec}(A)$, then the equilibrium solution $x(k) \equiv x_e$ of the discrete-time non-linear dynamical system* (10) *is asymptotically stable.*

(ii) *If there exists $\lambda \in \operatorname{spec}(A)$ such that $|\lambda| > 1$, then the equilibrium solution $x(k) \equiv x_e$ of the discrete-time non-linear dynamical system* (10) *is unstable.*

(iii) *Let $x_e = 0$, let $|\lambda| < 1$, where $\lambda \in \operatorname{spec}(A)$, let $p >> 0$ be such that $A^{\mathrm{T}} p << p$, and define $\mathcal{D}_A \triangleq \{x \in \overline{\mathbb{R}}^n_+ : p^{\mathrm{T}} x < \gamma\}$, where $\gamma \triangleq \sup\{\varepsilon > 0: p^{\mathrm{T}} f(x) < p^{\mathrm{T}} x,\ x \in \overline{\mathbb{R}}^n_+,\ \|x\| < \varepsilon\}$ and $\|x\| \triangleq \sum_{i=1}^n p_i |x_i|$. Then $\mathcal{D}_A$ is a subset of the domain of attraction for* (10).

Proof: (i) and (ii) are restatements of Lyapunov's indirect method (Khalil 1996) as applied to discrete-time non-linear non-negative dynamical systems. To prove (iii) note that it follows from Lemma 2 that if $f: \mathcal{D} \to \mathbb{R}^n$ is non-negative then A is non-negative. Hence, since $|\lambda| < 1$, where $\lambda \in \operatorname{spec}(A)$, it follows from (vi) of Theorem 1 that there exists $p >> 0$ such that $A^{\mathrm{T}} p << p$. Now, using the linear Lyapunov function candidate $V(x) = p^{\mathrm{T}} x$, it follows that the closed subset $\mathcal{D}_A$ of $\overline{\mathbb{R}}^n_+$ is a subset of the domain of attraction for (10) since $\Delta V(x) < 0$ for all $x \in \mathcal{D}_A \backslash \{0\} \subseteq \overline{\mathbb{R}}^n_+ \backslash \{0\}$. $\square$

Next, we show that discrete-time non-linear compartmental dynamical systems are a special case of discrete-time non-linear non-negative dynamical systems. To see this, once again let $x_i(k)$, $k \in \mathcal{N}$, $i = 1,\dots,n$, denote the mass (and hence a non-negative quantity) of the ith subsystem of the compartmental system shown in figure 1 with $a_{ij} x_j(k)$ replaced by $\hat{a}_{ij}(x(k))$ for $i,j = 1,\dots,n$, let $\hat{a}_{ii}(x) \geq 0$, $x \in \overline{\mathbb{R}}^n_+$, denote the average flow of material loss of the ith subsystem over the discretized interval h, let $w_i(k) \geq 0$, $k \in \mathcal{N}$, $i = 1,\dots,n$, denote the mass inflow supplied to the ith subsystem, and let $\phi_{ij}(x(k))$, $k \in \mathcal{N}$, $i \neq j$, $i,j = 1,\dots,n$, denote the net mass flow from the jth subsystem to the ith subsystem given by $\phi_{ij}(x(k)) = \hat{a}_{ij}(x(k)) - \hat{a}_{ji}(x(k))$, where the average (over the discretized interval h) rate of material flow $\hat{a}_{ij}(x) \geq 0$, $i \neq j$, $i,j = 1,\dots,n$. Hence, a mass balance for the whole compartmental model system with step $h = 1$ yields

$$\Delta x_i(k) = -\hat{a}_{ii}(x(k)) + \sum_{j=1,\, i\neq j}^{n} \left[\hat{a}_{ij}(x(k)) - \hat{a}_{ji}(x(k))\right] + w_i(k), \quad k \in \mathcal{N}, \quad i = 1,\dots,n \tag{11}$$

or, equivalently

$$x(k+1) = f(x(k)) + w(k), \quad x(0) = x_0, \quad k \in \mathcal{N} \tag{12}$$

where $x = [x_1,\dots,x_n]^{\mathrm{T}}$, $w = [w_1,\dots,w_n]^{\mathrm{T}}$, $f(x) = [f_1(x),\dots,f_n(x)]^{\mathrm{T}}$, and for $i = 1,\dots,n$, $f_i(x) = x_i - \hat{a}_{ii}(x) + \sum_{j=1,\, i\neq j}^{n} [\hat{a}_{ij}(x) - \hat{a}_{ji}(x)]$. Since all mass flows as well as compartment sizes are non-negative, it follows that for all $i = 1,\dots,n$, $f_i(x) \geq 0$ for all $x \in \overline{\mathbb{R}}^n_+$. The above physical constraints are implied by $\hat{a}_{ij}(x) \geq 0$, $\hat{a}_{ii}(x) \geq 0$ and $w_i \geq 0$, for all $i,j = 1,\dots,n$, and $x \in \overline{\mathbb{R}}^n_+$. The above physical constraints imply that f is non-negative. Taking the total mass of the compartmental system $V(x) = e^{\mathrm{T}} x = \sum_{i=1}^n x_i$ as a Lyapunov function for (12) (with $w(k) \equiv 0$) and assuming $\hat{a}_{ij}(0) = 0$, $i,j = 1,\dots,n$, it follows that

$$\Delta V(x) = \sum_{i=1}^{n} \Delta x_i = -\sum_{i=1}^{n} \hat{a}_{ii}(x) + \sum_{i=1}^{n} \sum_{j=1,\, i\neq j}^{n} \left[\hat{a}_{ij}(x) - \hat{a}_{ji}(x)\right] = -\sum_{i=1}^{n} \hat{a}_{ii}(x) \leq 0, \quad x \in \overline{\mathbb{R}}^n_+, \tag{13}$$

which shows that the zero solution $x(k) \equiv 0$ of the discrete-time non-linear inflow-closed ($w(k) \equiv 0$) compartmental system given by (12) is Lyapunov stable. If (12) with $w(k) \equiv 0$ has losses (outflows) from all compartments, then $\hat{a}_{ii}(x) > 0$, $x \in \overline{\mathbb{R}}^n_+$, $x \neq 0$, and by (13), the zero solution $x(k) \equiv 0$ to (12) (with $w(k) \equiv 0$) is asymptotically stable. As in the linear case, non-linear discrete-time compartmental systems are port-controlled Hamiltonian systems and hence conservative systems. This follows from the fact that (11) can be equivalently written as

$$\Delta x(k) = [J_n(x(k)) - D(x(k))]\left(\frac{\partial V}{\partial x}(x(k))\right)^{\mathrm{T}} + w(k), \quad x(0) = x_0, \quad k \in \mathcal{N}, \tag{14}$$

where $J_n(x)$ is a skew-symmetric matrix function with $J_{n(i,i)}(x) = 0$ and $J_{n(i,j)}(x) = \hat{a}_{ij}(x) - \hat{a}_{ji}(x)$, $i \neq j$, and $D(x) = \operatorname{diag}[\hat{a}_{11}(x), \hat{a}_{22}(x), \ldots, \hat{a}_{nn}(x)] \geq\geq 0$, $x \in \overline{\mathbb{R}}_+^n$.

Finally, if $f(x)\colon \overline{\mathbb{R}}_+^n \to \mathbb{R}^n$ is continuously differentiable and $x_e = 0$ so that $f(0) = 0$, then it follows that (see Jacquez and Simon (1993) for details) $\hat{a}_{ij}(x) = a_{ij}(x)x_j$, where the state-dependent transfer coefficients $a_{ij}(x) \geq 0$, $x \in \overline{\mathbb{R}}_+^n$, $i, j = 1, \ldots, n$, and $a_{ij}(\cdot)$ is continuous. In this case, equation (11) becomes

$$x_i(k+1) = x_i(k) - \left[a_{ii}(x(k)) + \sum_{j=1,\, i\neq j}^{n} a_{ji}(x(k))\right] x_i(k) + \sum_{j=1,\, i\neq j}^{n} a_{ij}(x(k))x_j(k) + w_i(k), \qquad k \in \mathcal{N} \tag{15}$$

for $i = 1, \ldots, n$, or, equivalently

$$x(k+1) = A(x(k))x(k) + w(k), \qquad x(0) = x_0, \quad k \in \mathcal{N} \tag{16}$$

where for $i, j = 1, \ldots, n$

$$A_{(i,j)}(x) = \begin{cases} 1 - \sum_{l=1}^{n} a_{li}(x), & i = j \\ a_{ij}(x), & i \neq j \end{cases} \tag{17}$$

Note that using identical arguments as in the linear case, $1 \geq \sum_{l=1}^{n} a_{li}(x)$, $x \in \overline{\mathbb{R}}_+^n$, and hence $A(x)$ is non-negative for all $x \in \overline{\mathbb{R}}_+^n$. Hence, once again using the total mass $V(x) = e^{\mathrm{T}}x$ as a Lyapunov function for (16) (with $w(k) \equiv 0$) it follows that

$$\Delta V(x) = e^{\mathrm{T}}\Delta x = e^{\mathrm{T}}[A(x) - I]x = -\sum_{i=1}^{n} a_{ii}(x)x_i \leq 0, \qquad x \in \overline{\mathbb{R}}_+^n \tag{18}$$

which shows that the zero solution $x(k) \equiv 0$ of the inflow-closed ($w(k) \equiv 0$) discrete-time system given by (16) is Lyapunov stable. In light of the above and (14) we have the following result on stability of solutions for discrete-time non-linear inflow-closed compartmental systems.

Theorem 6: *Consider the inflow-closed non-linear compartmental system given by* (14) *where* $V(x) = e^{\mathrm{T}}x$. *If* $J_n(0) = 0$, $D(x) \geq\geq 0$, $x \in \overline{\mathbb{R}}_+^n$, *then the zero solution* $x(k) \equiv 0$ *to* (14) (*with* $w(k) \equiv 0$) *is Lyapunov stable. If, in addition,* $D(x) > 0$, $x \in \overline{\mathbb{R}}_+^n \backslash \{0\}$, *and* $D(0) = 0$, *then the zero solution* $x(k) \equiv 0$ *to* (14) *is asymptotically stable.*

Proof: Lyapunov stability follows by noting that $V(x) = e^{\mathrm{T}}x$ is a Lyapunov function candidate for (14) and $\Delta V(x) = e^{\mathrm{T}}\Delta x e = e^{\mathrm{T}}(J_n(x) - D(x))e = -e^{\mathrm{T}}D(x)e \leq 0$, $x \in \overline{\mathbb{R}}_+^n$. To show asymptotic stability note that if $D(x) > 0$, $x \in \overline{\mathbb{R}}_+^n \backslash \{0\}$, then $\Delta V(x) = -e^{\mathrm{T}}D(x)e < 0$, $x \in \overline{\mathbb{R}}_+^n \backslash \{0\}$. □

Finally, we present necessary and sufficient conditions for *monotonicity* of non-linear non-negative dynamical systems. For this result we require the following definition.

Definition 6: Consider the non-linear dynamical system (10) where $f\colon \mathcal{D} \to \mathbb{R}^n$ is non-negative. The non-linear dynamical system (10) is *monotonic* if there exists a matrix $Q \in \mathbb{R}^{n\times n}$ such that $Q = \operatorname{diag}[q_1, \ldots, q_n]$, $q_i = \pm 1$, $i = 1, \ldots, n$, and for every $x_0 \in \overline{\mathbb{R}}_+^n$, $Qx(k_2) \leq\leq Qx(k_1)$, $0 \leq k_1 \leq k_2$.

Theorem 7: *The non-linear dynamical system* (10), *where* $f\colon \mathcal{D} \to \mathbb{R}^n$ *is non-negative and* $x_0 \in \overline{\mathbb{R}}_+^n$, *is monotonic if and only if there exists a matrix* $Q \in \mathbb{R}^{n\times n}$ *such that* $Q = \operatorname{diag}[q_1, \ldots, q_n]$, $q_i = \pm 1$, $i = 1, \ldots, n$ *and* $Qf(x) \leq\leq Qx$, $x \in \overline{\mathbb{R}}_+^n$.

Proof: To show sufficiency, assume there exists $Q = \operatorname{diag}[q_1, \ldots, q_n]$, $q_i = \pm 1$, $i = 1, \ldots, n$, such that $Qf(x) \leq\leq Qx$, $x \in \overline{\mathbb{R}}_+^n$. Now, it follows from (10) that

$$Qx(k+1) = Qf(x(k)), \qquad x(0) = x_0, \quad k \in \mathcal{N} \tag{19}$$

which further implies that for every $k \in \mathcal{N}$

$$Qx(k_2) = Qx(k_1) + \sum_{k=k_1}^{k_2-1} [Qf(x(k))] \tag{20}$$

Next, since $f(\cdot)$ is non-negative it follows from Proposition 2 that $x(k) \geq\geq 0$, $k \in \mathcal{N}$. Hence, since $Qf(x) \leq\leq Qx$, $x \in \overline{\mathbb{R}}_+^n$, it follows that $Qf(x(k)) \leq\leq Qx(k)$, $k \in \mathcal{N}$, which implies that for every $x_0 \in \overline{\mathbb{R}}_+^n$, $Qx(k_2) \leq\leq Qx(k_1)$, $0 \leq k_1 \leq k_2$.

To show necessity, assume that (10) is monotonic. Now, suppose, *ad absurdum*, there exist $J \in \{1, \ldots, n\}$ and $x_0 \in \overline{\mathbb{R}}_+^n$ such that $[Qf(x_0)]_J > [Qx_0]_J$. Hence, it follows from (20) that $[Qx(1)]_J = [Qx_0]_J + [Qf(x_0) - Qx_0]_J > [Qx_0]_J$, which is a contradiction. Hence, $Qf(x) \leq\leq Qx$, $x \in \overline{\mathbb{R}}_+^n$. □

As mentioned in the introduction, it is of interest to determine sufficient conditions under which masses/concentrations for non-linear compartmental systems are Lyapunov stable and convergent, guaranteeing the absence of limit cycling behaviour. The following result follows from Theorem 7 and provides sufficient conditions for the absence of limit cycles in non-linear compartmental systems.

Theorem 8: *Consider the non-linear compartmental dynamical system* (14) *with* $w(k) \equiv 0$. *If there exists a matrix* $Q \in \mathbb{R}^{n\times n}$ *such that* $Q = \operatorname{diag}[q_1, \ldots, q_n]$, $q_i = \pm 1$, $i = 1, \ldots, n$, *and* $Qf(x) \leq\leq Qx$, $x \in \overline{\mathbb{R}}_+^n$, *then for every* $x_0 \in \overline{\mathbb{R}}_+^n$, $\lim_{k\to\infty} x(k)$ *exists.*

Proof: Let $V(x) = e^{\mathrm{T}}x$, $x \in \overline{\mathbb{R}}^n_+$. Now, it follows from (13) that $\Delta V(x(k)) \le 0$, $k \in \mathcal{N}$, where $x(k)$, $k \in \mathcal{N}$, denotes the solution of (14), which implies that $V(x(k)) \le V(x_0) = e^{\mathrm{T}}x_0$, $k \in \mathcal{N}$, and hence for every $x_0 \in \overline{\mathbb{R}}^n_+$, the solution $x(k)$, $k \in \mathcal{N}$, of (14) is bounded. Hence, for every $i \in \{1, \ldots, n\}$, $x_i(k)$, $k \in \mathcal{N}$, is bounded. Furthermore, it follows from Theorem 7 that $x_i(k)$, $k \in \mathcal{N}$, is monotonic. Now, since $x_i(\cdot)$, $i \in \{1, \ldots, n\}$, is bounded and monotonic, it follows that $\lim_{k\to\infty} x_i(k)$, $i \in \{1, \ldots, n\}$, exists. Hence, $\lim_{k\to\infty} x(k)$ exists. □

5. Dissipativity theory for discrete-time non-negative dynamical systems

In this section we extend the notion of dissipativity to discrete-time non-linear non-negative dynamical systems. Specifically, we consider discrete-time dynamical systems $\mathcal{G}$ of the form

$$x(k+1) = f(x(k)) + G(x(k))u(k), \qquad x(0) = x_0, \quad k \in \mathcal{N} \tag{21}$$

$$y(k) = h(x(k)) + J(x(k))u(k) \tag{22}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $f: \mathbb{R}^n \to \mathbb{R}^n$, $G: \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h: \mathbb{R}^n \to \mathbb{R}^l$, and $J: \mathbb{R}^n \to \mathbb{R}^{l\times m}$. We assume that $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$ are continuous mappings and $f(x_e) = x_e$ and $h(x_e) = x_e$. For simplicity of exposition here we assume $x_e = 0$.† First, we provide a key definition and several results concerning dynamical systems of the form (21) and (22) with non-negative inputs and non-negative outputs.

Definition 7: The non-linear dynamical system $\mathcal{G}$ given by (21) and (22) is *non-negative* if for every $x(0) \in \overline{\mathbb{R}}^n_+$ and $u(k) \ge\ge 0$, $k \in \mathcal{N}$, the solution $x(k)$, $k \in \mathcal{N}$, to (21) and the output $y(k)$, $k \in \mathcal{N}$, are non-negative.

Proposition 3: *Consider the non-linear dynamical system $\mathcal{G}$ given by* (21) *and* (22). *If $f: \mathcal{D} \to \mathbb{R}^n$ is non-negative*, $G(x) \ge\ge 0$, $h(x) \ge\ge 0$ *and* $J(x) \ge\ge 0$, $x \in \overline{\mathbb{R}}^n_+$, *then $\mathcal{G}$ is non-negative*.

Proof: Let $x(0) \in \overline{\mathbb{R}}^n_+$, $u(k) \equiv 0$, and suppose $f: \mathcal{D} \to \mathbb{R}^n$ is non-negative and $G(x) \ge\ge 0$, $x \in \overline{\mathbb{R}}^n_+$. For every $i \in \{1, \ldots, n\}$ it follows that $x_i(k+1) = f_i(x(k)) \ge 0$. Hence, $x(k+1) = f(x(k)) \ge\ge 0$, $k \in \mathcal{N}$. Furthermore, for $u(k) \ge\ge 0$, $k \in \mathcal{N}$, it follows that $x(k+1) = f(x(k)) + G(x(k))u(k) \ge\ge 0$, $k \in \mathcal{N}$. Thus, $x(k) \in \overline{\mathbb{R}}^n_+$, $k \in \mathcal{N}$. Using identical arguments, in the case where $h(x) \ge\ge 0$ and $J(x) \ge\ge 0$, $x \in \overline{\mathbb{R}}^n_+$, it follows that $y(k) \in \overline{\mathbb{R}}^m_+$, $k \in \mathcal{N}$. Hence, $\mathcal{G}$ is non-negative. □

For the dynamical system $\mathcal{G}$ given by (21) and (22) a function $s: \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ such that $s(0,0) = 0$ is called a *supply rate* (Willems 1972 a) if it is locally summable; that is, for all input–output pairs $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $s(\cdot,\cdot)$ satisfies $\sum_{i=k_1}^{k_2} |s(u(i), y(i))| < \infty$, $k_1, k_2 \in \mathcal{N}$. For the remainder of the results of this paper we assume that $f(x) \ge\ge 0$, $G(x) \ge\ge 0$, $h(x) \ge\ge 0$ and $J(x) \ge\ge 0$, $x \in \overline{\mathbb{R}}^n_+$. The following definition introduces the notion of dissipativity for a discrete-time non-negative dynamical system.

Definition 8: The non-negative dynamical system $\mathcal{G}$ given by (21) and (22) is *geometrically dissipative* (resp., *dissipative*) *with respect to the supply rate* $s: \overline{\mathbb{R}}^m_+ \times \overline{\mathbb{R}}^l_+ \to \mathbb{R}$ if there exists a continuous non-negative-definite function $V_s: \overline{\mathbb{R}}^n_+ \to \overline{\mathbb{R}}_+$ called a *storage function* and a scalar $\rho > 1$ (resp., $\rho = 1$) such that $V_s(0) = 0$ and the *dissipation inequality*

$$\rho^k V_s(x(k)) \le \rho^{k_0} V_s(x(k_0)) + \sum_{i=k_0}^{k-1} \rho^i s(u(i), y(i)), \qquad k \ge k_0 \tag{23}$$

is satisfied for all $k_0, k \in \mathcal{N}$, where $x(k)$, $k \ge k_0$, is the solution of (21) with $u \in \overline{\mathbb{R}}^m_+$. The non-negative dynamical system $\mathcal{G}$ given by (21) and (22) is *lossless with respect to the supply rate* $s: \overline{\mathbb{R}}^m_+ \times \overline{\mathbb{R}}^l_+ \to \mathbb{R}$ if the dissipation inequality (23) is satisfied as an equality with $\rho = 1$ for all $k \ge k_0$.

Remark 2: An equivalent statement for geometric dissipativity of a discrete-time non-negative dynamical system $\mathcal{G}$ is

$$\rho V_s(x(k+1)) - V_s(x(k)) \le s(u(k), y(k)), \\ k \in \mathcal{N}, \qquad u \in \overline{\mathbb{R}}^m_+, \qquad y \in \overline{\mathbb{R}}^l_+ \tag{24}$$

Since discrete-time non-negative dynamical systems are a subset of discrete-time dynamical systems, standard discrete-time dissipativity theory (Chellaboina and Haddad 2002) with *quadratic* storage functions and *quadratic* supply rates involving Kalman–Yakubovich–Popov conditions also holds for discrete-time non-negative dynamical systems. In this paper, however, motivated by conservation of mass laws, we develop dissipativity notions for discrete-time non-negative dynamical systems with respect to *linear* supply rates.

The following result presents Kalman–Yakubovich–Popov conditions for discrete-time non-negative dynamical systems with linear supply rates of the form $s(u,y) = q^{\mathrm{T}}y + r^{\mathrm{T}}u$, where $q \in \mathbb{R}^l$, $q \ne 0$, and

† In the case where $\mathcal{G}$ is non-negative, this assumption is *not* without loss of generality since shifting the equilibrium can destroy the non-negativity of the vector field f and the non-negativity of h. However, using minor modifications in the proofs of the theorems in this section, the results of this section also hold for the case where $x_e \ne 0$.

$r \in \mathbb{R}^m$, $r \neq 0$. First, however, the following definition is required.

Definition 9: A discrete-time non-negative dynamical system $\mathcal{G}$ is *zero-state observable* if for all $x \in \overline{\mathbb{R}}_+^n$, $u(k) \equiv 0$ and $y(k) \equiv 0$ implies $x(k) \equiv 0$. A discrete-time non-negative dynamical system $\mathcal{G}$ is *completely reachable* if for all $x_0 \in \overline{\mathbb{R}}_+^n$, there exist a $k_\mathrm{i} \leq k_0$, and square summable input $u(k)$ defined on $[k_\mathrm{i}, k_0]$, such that the state $x(k)$, $k \geq k_\mathrm{i}$, can be driven from $x(k_\mathrm{i}) = 0$ to $x(k_0) = x_0$.

Theorem 9: *Let $q \in \mathbb{R}^l$ and $r \in \mathbb{R}^m$. Consider the nonlinear non-negative dynamical system $\mathcal{G}$ given by* (21) *and* (22) *where $f: \mathcal{D} \to \mathbb{R}^n$ is non-negative, $G(x) \geq\geq 0$, $h(x) \geq\geq 0$ and $J(x) \geq\geq 0$, $x \in \overline{\mathbb{R}}_+^n$. If there exist functions $V_\mathrm{s}: \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$, $\ell: \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$, $\mathcal{W}: \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+^m$, $P_{1u}: \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and a scalar $\rho > 1$ (resp., $\rho = 1$) such that $V_\mathrm{s}(\cdot)$ is continuous and non-negative definite, $V_\mathrm{s}(0) = 0$*

$$V_\mathrm{s}(f(x)+G(x)u) = V_\mathrm{s}(f(x)) + P_{1u}(x)u, \quad x \in \overline{\mathbb{R}}_+^n, \quad u \in \overline{\mathbb{R}}_+^m \tag{25}$$

and, for all $x \in \overline{\mathbb{R}}_+^n$

$$0 = V_\mathrm{s}(f(x)) - \frac{1}{\rho}V_\mathrm{s}(x) - q^\mathrm{T}h(x) + \ell(x) \tag{26}$$

$$0 = P_{1u}(x) - q^\mathrm{T}J(x) - r^\mathrm{T} + \mathcal{W}^\mathrm{T}(x) \tag{27}$$

then $\mathcal{G}$ is geometrically dissipative (resp., dissipative) with respect to the supply rate $s(u, y) = q^\mathrm{T}y + r^\mathrm{T}u$.

Proof: Suppose that there exist functions $V_\mathrm{s}: \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$, $\ell: \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$, $\mathcal{W}: \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+^m$, $P_{1u}: \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and a scalar $\rho > 1$ (resp., $\rho = 1$), such that $V_\mathrm{s}(\cdot)$ is continuous, $V_\mathrm{s}(0) = 0$, and (25)–(27) are satisfied. Then, with $\hat{V}_\mathrm{s}(x) \triangleq (1/\rho)V_\mathrm{s}(x)$, it follows that for all $x \in \overline{\mathbb{R}}_+^n$ and $u \in \overline{\mathbb{R}}_+^m$

$$\begin{aligned}\rho\hat{V}_\mathrm{s}(f(x) + G(x)u) - \hat{V}_\mathrm{s}(x) &= V_\mathrm{s}(f(x) + G(x)u) - \frac{1}{\rho}V_\mathrm{s}(x)\\ &= V_\mathrm{s}(f(x)) + P_{1u}(x)u - \frac{1}{\rho}V_\mathrm{s}(x)\\ &= q^\mathrm{T}h(x) - \ell(x) + q^\mathrm{T}J(x)u\\ &\quad + r^\mathrm{T}u - \mathcal{W}^\mathrm{T}(x)u\\ &\leq q^\mathrm{T}y + r^\mathrm{T}u\end{aligned}$$

which implies that $\mathcal{G}$ is geometrically dissipative (resp., dissipative) with respect to the supply rate $s(u, y) = q^\mathrm{T}y + r^\mathrm{T}u$. □

Remark 3: As in standard dissipativity theory with quadratic supply rates (Hill and Moylan 1976), the concepts of linear supply rates and linear and nonlinear storage functions provide a generalized mass and energy balance interpretation. Specifically, using (25)–(27) with $\rho = 1$ it follows that

$$\sum_{\kappa=k_0}^{k-1}\left[q^\mathrm{T}y(\kappa)+r^\mathrm{T}u(\kappa)\right] = V_\mathrm{s}(x(k)) - V_\mathrm{s}(x(k_0)) + \sum_{\kappa=k_0}^{k-1}\left[\ell^\mathrm{T}(x(\kappa))x(\kappa)+\mathcal{W}^\mathrm{T}(x(\kappa))u(\kappa)\right] \tag{28}$$

which can be interpreted as a generalized mass balance equation where $V_\mathrm{s}(x(k)) - V_\mathrm{s}(x(k_0))$ is the stored mass of the discrete-time non-negative system and the sum on the right corresponds to the expelled mass of the non-negative system. Rewriting (28) as

$$\begin{aligned}\Delta V_\mathrm{s}(x) &= V_\mathrm{s}(f(x)) - V_\mathrm{s}(x)\\ &= q^\mathrm{T}y + r^\mathrm{T}u - \left[\ell^\mathrm{T}(x)x + \mathcal{W}^\mathrm{T}(x)u\right]\end{aligned} \tag{29}$$

yields a mass conservation equation which shows that the system mass transport is equal to the supplied system mass minus the expelled system mass.

Remark 4: Recall that in standard dissipativity theory if $\mathcal{G}$ is zero-state observable and there exists a function $\kappa: \mathbb{R}^l \to \mathbb{R}^m$ such that $s(\kappa(y), y) < 0$, $y \neq 0$, then the storage function $V_\mathrm{s}(\cdot)$ satisfies $V_\mathrm{s}(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$ (Hill and Moylan 1976). Similarly, for the discrete-time non-negative dynamical system $\mathcal{G}$, it can be shown that if $\mathcal{G}$ is zero-state observable and there exists a function $\kappa: \overline{\mathbb{R}}_+^l \to \overline{\mathbb{R}}_+^m$ such that $s(\kappa(y), y) < 0$, $y \in \overline{\mathbb{R}}_+^l$, $y \neq 0$, then $V_\mathrm{s}(x) > 0$, $x \in \overline{\mathbb{R}}_+^n$, $x \neq 0$. Hence, in the case of a linear supply rate, there always exists a matrix $K \in \mathbb{R}^{m\times l}$ such that $q + K^\mathrm{T}r << 0$ which implies that if $\mathcal{G}$ is zero-state observable, then $V_\mathrm{s}(x) > 0$, $x \in \overline{\mathbb{R}}_+^n$, $x \neq 0$.

Remark 5: Note that if a discrete-time non-negative dynamical system $\mathcal{G}$ is zero-state observable and dissipative with respect to the linear supply rate $s(u, y) = q^\mathrm{T}y + r^\mathrm{T}u$, and if $q \leq\leq 0$ and $u \equiv 0$, it follows that $\Delta V_\mathrm{s}(x(k)) \leq q^\mathrm{T}y(k) \leq 0$, $k \in \mathcal{N}$. Hence, the undisturbed ($u(k) \equiv 0$) system $\mathcal{G}$ is Lyapunov stable. Alternatively, if a discrete-time non-negative dynamical system $\mathcal{G}$ is zero-state observable and geometrically dissipative with respect to the linear supply rate $s(u, y) = q^\mathrm{T}y + r^\mathrm{T}u$, and if $q \leq\leq 0$ and $u \equiv 0$, it follows that

$$\Delta V_\mathrm{s}(x(k)) \leq -\frac{\rho - 1}{\rho}V_\mathrm{s}(x(k)) + \frac{1}{\rho}q^\mathrm{T}y(k)$$

$k \in \mathcal{N}$ and $\rho > 1$. Hence, the undisturbed ($u(k) \equiv 0$) system $\mathcal{G}$ is asymptotically stable.

Next, we provide necessary and sufficient conditions for the case where $\mathcal{G}$ given by (21) and (22) is lossless with respect to the linear supply rate $s(u, y) = q^\mathrm{T}y + r^\mathrm{T}u$.

Theorem 10: *Let $q \in \mathbb{R}^l$ and $r \in \mathbb{R}^m$. Consider the nonlinear non-negative dynamical system $\mathcal{G}$ given by* (21) *and* (22) *where $f\colon \mathcal{D} \to \mathbb{R}^n$ is non-negative, $G(x) \geq\geq 0$, $h(x) \geq\geq 0$ and $J(x) \geq\geq 0$, $x \in \overline{\mathbb{R}}^n_+$. Then $\mathcal{G}$ is lossless with respect to the supply rate $s(u,y) = q^{\mathrm{T}}y + r^{\mathrm{T}}u$, $u \in \overline{\mathbb{R}}^m_+$, if and only if there exist functions $V_{\mathrm{s}}\colon \overline{\mathbb{R}}^n_+ \to \overline{\mathbb{R}}_+$ and $P_{1u}\colon \mathbb{R}^n \to \mathbb{R}^{1\times m}$ such that $V_{\mathrm{s}}(\cdot)$ is continuous, $V_{\mathrm{s}}(0) = 0$, and for all $x \in \overline{\mathbb{R}}^n_+$, equation* (25) *holds and*

$$0 = V_{\mathrm{s}}(f(x)) - V_{\mathrm{s}}(x) - q^{\mathrm{T}}h(x) \tag{30}$$

$$0 = P_{1u}(x) - q^{\mathrm{T}}J(x) - r^{\mathrm{T}} \tag{31}$$

If, in addition, $V_{\mathrm{s}}(\cdot)$ is continuously differentiable, then

$$P_{1u}(x) = V_{\mathrm{s}}'(f(x))G(x) \tag{32}$$

Proof: Sufficiency follows as in the proof of Theorem 9. To show necessity, suppose that $\mathcal{G}$ is lossless with respect to the linear supply rate $s(u,y) = q^{\mathrm{T}}y + r^{\mathrm{T}}u$. Then, it follows that there exists a continuous function $V_{\mathrm{s}}\colon \overline{\mathbb{R}}^n_+ \to \overline{\mathbb{R}}_+$ such that

$$\begin{aligned} V_{\mathrm{s}}(f(x) + G(x)u) &= V_{\mathrm{s}}(x) + s(u,y) \\ &= V_{\mathrm{s}}(x) + q^{\mathrm{T}}y + r^{\mathrm{T}}u \\ &= V_{\mathrm{s}}(x) + q^{\mathrm{T}}h(x) + (q^{\mathrm{T}}J(x) + r^{\mathrm{T}})u, \\ & \qquad x \in \overline{\mathbb{R}}^n_+, \qquad u \in \overline{\mathbb{R}}^m_+ \end{aligned} \tag{33}$$

Since the right-hand side of (33) is linear in u it follows that $V_{\mathrm{s}}(f(x) + G(x)u)$ is linear in u and hence there exists $P_{1u}\colon \mathbb{R}^n \to \mathbb{R}^{1\times m}$ such that

$$V_{\mathrm{s}}(f(x)+G(x)u) = V_{\mathrm{s}}(f(x)) + P_{1u}(x)u, \quad x \in \overline{\mathbb{R}}^n_+, \qquad u \in \overline{\mathbb{R}}^m_+ \tag{34}$$

Now, using (34) and equating coefficients of equal powers in (33) yields (30) and (31).

Finally, if $V_{\mathrm{s}}(\cdot)$ is continuously differentiable, applying a Taylor series expansion on (34) about $u=0$ yields

$$P_{1u}(x) = \left.\frac{\partial V_{\mathrm{s}}(f(x) + G(x)u)}{\partial u}\right|_{u=0} = V_{\mathrm{s}}'(f(x))G(x) \tag{35}$$

□

Next, we provide a key definition for discrete-time non-negative dynamical systems which are dissipative with respect to a very special supply rate.

Definition 10: A non-negative dynamical system $\mathcal{G}$ of the form (21) and (22) is *non-accumulative* (resp., *geometrically non-accumulative*) if $\mathcal{G}$ is dissipative (resp., geometrically dissipative) with respect to the supply rate $s(u,y) = e^{\mathrm{T}}u - e^{\mathrm{T}}y$.

If $\mathcal{G}$ is non-accumulative, then it follows from (24) that $\Delta V_{\mathrm{s}}(x(k)) \leq e^{\mathrm{T}}u(k) - e^{\mathrm{T}}y(k)$, $k \in \mathcal{N}$, where $u \in \overline{\mathbb{R}}^m_+$ and $y \in \overline{\mathbb{R}}^l_+$. If the components $u_i(\cdot)$, $i = 1,\ldots,m$, of $u(\cdot)$ denote mass inputs to the system $\mathcal{G}$ and the components $y_i(\cdot)$, $i = 1,\ldots,l$, of $y(\cdot)$ denote the mass outputs of the system $\mathcal{G}$, then dissipativity with respect to the linear supply rate $s(u,y) = e^{\mathrm{T}}u - e^{\mathrm{T}}y$ implies that the change in system mass is always less than or equal to the difference between the system mass input and system mass output.

All discrete-time compartmental systems with measured outputs corresponding to material outflows are non-accumulative. To see this, consider (14) with storage function $V_{\mathrm{s}}(x) = e^{\mathrm{T}}x$ and outputs y corresponding to a partial observation of the material outflows. Specifically, without loss of generality, let y correspond to the first l outflows so that $y = [\hat{a}_{11}(x)x_1,\ \hat{a}_{22}(x)x_2,\ldots,\ \hat{a}_{ll}(x)x_l]$. Now, note that

$$y = D(x)\left(\frac{\partial V}{\partial x}\right)^{\mathrm{T}} - D_{\mathrm{r}}(x)\left(\frac{\partial V}{\partial x}\right)^{\mathrm{T}}$$

where $D_{\mathrm{r}}(x) = \operatorname{diag}[0,\ldots,0,\hat{a}_{l+1l+1}(x)x_{l+1},\ldots,\ \hat{a}_{nn}(x)x_n] \geq\geq 0$, $x \in \overline{\mathbb{R}}^n_+$. Hence

$$\begin{aligned} \Delta V_{\mathrm{s}}(x) &= e^{\mathrm{T}}\left[[J_n(x) - D(x)]\left(\frac{\partial V}{\partial x}\right)^{\mathrm{T}} + w\right] \\ &= e^{\mathrm{T}}w - e^{\mathrm{T}}y - \left(\frac{\partial V}{\partial x}\right)D_{\mathrm{r}}(x)\left(\frac{\partial V}{\partial x}\right)^{\mathrm{T}} \\ &\leq e^{\mathrm{T}}w - e^{\mathrm{T}}y, \quad x \in \overline{\mathbb{R}}^n_+ \end{aligned} \tag{36}$$

Note that in the case where the system is closed, $\Delta V_{\mathrm{s}}(x) = 0$, $x \in \overline{\mathbb{R}}^n_+$, which corresponds to conservation of mass in the system.

Finally, we present a key result on linearization of discrete-time non-negative dissipative dynamical systems. For this result we assume that the storage function $V_{\mathrm{s}}(\cdot)$ belongs to C^2.

Theorem 11: *Let $q \in \mathbb{R}^l$ and $r \in \mathbb{R}^m$ and assume $\mathcal{G}$ given by* (21) *and* (22) *is such that $f\colon \mathcal{D} \to \mathbb{R}^n$ is non-negative, $G(x) \geq\geq 0$, $h(x) \geq\geq 0$ and $J(x) \geq\geq 0$, $x \in \overline{\mathbb{R}}^n_+$. Suppose $\mathcal{G}$ is geometrically dissipative (resp., dissipative) with respect to the supply rate $s(u,y) = q^{\mathrm{T}}y + r^{\mathrm{T}}u$. Then, there exist $p \in \overline{\mathbb{R}}^n_+$, $l \in \overline{\mathbb{R}}^n_+$, and $w \in \overline{\mathbb{R}}^m_+$ and a scalar $\rho > 1$ (resp., $\rho = 1$) such that*

$$0 = A^{\mathrm{T}}p - \frac{1}{\rho}p - C^{\mathrm{T}}q + l \tag{37}$$

$$0 = B^{\mathrm{T}}p - D^{\mathrm{T}}q - r + w \tag{38}$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}, \quad B = G(0), \quad C = \left.\frac{\partial h}{\partial x}\right|_{x=0}, \quad D = J(0) \tag{39}$$

If, in addition, (A, C) is observable, then $p >> 0$.

Proof: Assume that $\mathcal{G}$ is geometrically dissipative (resp., dissipative) with respect to the supply rate $s(u,y) = q^{\mathrm{T}}y + r^{\mathrm{T}}u$. Then, it follows that there exists a continuous function $V_{\mathrm{s}}: \mathbb{R}_+^n \to \mathbb{R}_+$ and a scalar $\rho > 1$ (resp., $\rho = 1$) such that

$$\rho V_{\mathrm{s}}(f(x) + G(x)u) - V_{\mathrm{s}}(x) \le q^{\mathrm{T}}y + r^{\mathrm{T}}u, \quad x \in \overline{\mathbb{R}}_+^n, \quad u \in \overline{\mathbb{R}}_+^m \tag{40}$$

Now, it follows from (40) that there exists a function $d: \overline{\mathbb{R}}_+^n \times \overline{\mathbb{R}}_+^m \to \overline{\mathbb{R}}_+$ such that $d(x,u) \ge 0$, $d(0,0) = 0$, and

$$0 = \rho V_{\mathrm{s}}(f(x) + G(x)u) - V_{\mathrm{s}}(x) - q^{\mathrm{T}}y - r^{\mathrm{T}}u + d(x,u), \quad x \in \overline{\mathbb{R}}_+^n, \quad u \in \overline{\mathbb{R}}_+^m \tag{41}$$

Next, expanding $V_{\mathrm{s}}(\cdot)$ and $d(\cdot,\cdot)$ via a Taylor series expansion about $x = 0$, $u = 0$, and using the fact that $V_{\mathrm{s}}(\cdot)$ and $d(\cdot,\cdot)$ are non-negative definite and $V_{\mathrm{s}}(0) = 0$, $d(0,0) = 0$, it follows that there exist $p \in \overline{\mathbb{R}}_+^n$, $l \in \overline{\mathbb{R}}_+^n$, and $w \in \overline{\mathbb{R}}_+^m$ such that $V_{\mathrm{s}}(x) = (1/\rho)p^{\mathrm{T}}x + V_{\mathrm{sr}}(x)$ and $d(x,u) = l^{\mathrm{T}}x + w^{\mathrm{T}}u + d_{\mathrm{sr}}(x,u)$, where $V_{\mathrm{sr}}: \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ and $d_{\mathrm{sr}}: \overline{\mathbb{R}}_+^n \times \overline{\mathbb{R}}_+^m \to \overline{\mathbb{R}}_+$ contain higher-order terms of $V_{\mathrm{s}}(\cdot)$ and $d(\cdot,\cdot)$, respectively. Next, let $f(x) = Ax + f_{\mathrm{r}}(x)$ and $h(x) = Cx + h_{\mathrm{r}}(x)$, where $f_{\mathrm{r}}(x)$ and $h_{\mathrm{r}}(x)$ contain the non-linear terms of $f(x)$ and $h(x)$, respectively, and let $G(x) = B + G_{\mathrm{r}}(x)$ and $J(x) = D + J_{\mathrm{r}}(x)$, where $G_{\mathrm{r}}(x)$ and $J_{\mathrm{r}}(x)$ contain the non-constant terms of $G(x)$ and $J(x)$ respectively. Using the above expressions (41) can be written as

$$0 = p^{\mathrm{T}}Ax + p^{\mathrm{T}}Bu - \frac{1}{\rho}p^{\mathrm{T}}x - q^{\mathrm{T}}(Cx + Du) - r^{\mathrm{T}}u + l^{\mathrm{T}}x + w^{\mathrm{T}}u + \delta(x,u) \tag{42}$$

where $\delta(x,u)$ is such that $\delta(x,u)/(\|x\| + \|u\|) \to 0$ as $\|x\| + \|u\| \to 0$.

Now, setting $u = 0$ in (42) and equating coefficients of equal powers yields (37). Alternatively, setting $x = 0$ in (42) and equating coefficients of equal powers yields (38).

Finally, to show that $p >> 0$ in the case where (A, C) is observable, note that it follows from Theorem 9 that the linearized system $\mathcal{G}$ with storage function $V_{\mathrm{s}}(x) = p^{\mathrm{T}}x$ is geometrically dissipative (resp., dissipative) with respect to the linear supply rate $s(u,y) = q^{\mathrm{T}}y + r^{\mathrm{T}}u$. Now, it follows from Remark 4 that $p >> 0$. □

6. Specialization to discrete-time linear non-negative dynamical systems

In this section we specialize the results of §5 to the case of linear discrete-time non-negative dynamical systems. Specifically, setting $f(x) = Ax$, $G(x) = B$, $h(x) = Cx$ and $J(x) = D$, the non-linear non-negative dynamical system given by (21) and (22) specializes to

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0, \quad k \in \mathcal{N} \tag{43}$$

$$y(k) = Cx(k) + Du(k) \tag{44}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$ and $D \in \mathbb{R}^{l\times m}$. Before providing linear dissipativity specializations, we present a key result on linear non-negative systems in the case where $u(k) \ge\ge 0$ and $y(k) \ge\ge 0$, $k \in \mathcal{N}$.

Theorem 12: *The linear dynamical system $\mathcal{G}$ given by* (43) *and* (44) *is non-negative if and only if $A \ge\ge 0$, $B \ge\ge 0$, $C \ge\ge 0$, and $D \ge\ge 0$.*

Proof: See Farina and Rinaldi (2000, p. 14). □

The following result presents necessary and sufficient Kalman–Yakubovich–Popov conditions for linear discrete-time non-negative dynamical systems with linear supply rates of the form $s(u,y) = q^{\mathrm{T}}y + r^{\mathrm{T}}u$, where $q \in \mathbb{R}^l$, $q \neq 0$, and $r \in \mathbb{R}^m$, $r \neq 0$. Note that for a discrete-time linear dynamical system to be dissipative with respect to a linear supply rate it is necessary that the storage function is also linear. However, since all storage functions are non-negative by definition, it follows that a storage function is non-negative if and only if there exists a linear transformation such that the discrete-time linear dynamical system is non-negative in a transformed basis. Hence, dissipativity theory of discrete-time linear dynamical systems with respect to linear supply rates is complete if we restrict our consideration to the class of discrete-time non-negative dynamical systems.

Theorem 13: *Let $q \in \mathbb{R}^l$ and $r \in \mathbb{R}^m$. Consider the non-negative dynamical system $\mathcal{G}$ given by* (43) *and* (44). *Then $\mathcal{G}$ is geometrically dissipative (resp., dissipative) with respect to the supply rate $s(u,y) = q^{\mathrm{T}}y + r^{\mathrm{T}}u$ if and only if there exist $p \in \overline{\mathbb{R}}_+^n$, $l \in \overline{\mathbb{R}}_+^n$, and $w \in \overline{\mathbb{R}}_+^m$, and a scalar $\rho > 1$ (resp., $\rho = 1$) such that*

$$0 = A^{\mathrm{T}}p - \frac{1}{\rho}p - C^{\mathrm{T}}q + l \tag{45}$$

$$0 = B^{\mathrm{T}}p - D^{\mathrm{T}}q - r + w \tag{46}$$

Proof: First, note that since $\mathcal{G}$ is non-negative it follows from Theorem 12 that $A \ge\ge 0$, $B \ge\ge 0$, $C \ge\ge 0$ and $D \ge\ge 0$. Sufficiency follows from Theorem 9 with $f(x) = Ax$, $G(x) = B$, $h(x) = Cx$, $J(x) = D$ and $V_{\mathrm{s}}(x) = p^{\mathrm{T}}x$. To show necessity, note that if the linear non-negative dynamical system (43) and (44) is dissipative with respect to the linear supply rate $s(u,y) = q^{\mathrm{T}}y + r^{\mathrm{T}}u$, then it follows from Theorem 11 with $f(x) = Ax$, $G(x) = B$, $h(x) = Cx$ and $J(x) = D$ that there exist $p \in \overline{\mathbb{R}}_+^n$, $l \in \overline{\mathbb{R}}_+^n$, and $w \in \overline{\mathbb{R}}_+^m$ such that (45) and (46) are satisfied. □

Remark 6: For a given $l \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, note that there exists $p \in \mathbb{R}^n$ such that (45) and (46) are satisfied if and only if $\operatorname{rank}[M \ \ y] = \operatorname{rank} M$, where

$$M \triangleq \begin{bmatrix} (A-(1/\rho)I_n)^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix}, \qquad y \triangleq \begin{bmatrix} C^{\mathrm{T}}q - l \\ D^{\mathrm{T}}q + r - w \end{bmatrix}$$

Now, there exist $p \geq\geq 0$, $l \geq\geq 0$ and $w \geq\geq 0$ such that (45) and (46) are satisfied if and only if the inequalities $p \geq\geq 0$ and $z - Mp \geq\geq 0$, where

$$z \triangleq \begin{bmatrix} C^{\mathrm{T}}q \\ D^{\mathrm{T}}q + r \end{bmatrix}$$

are satisfied. These equations comprise a set of $2n+m$ linear inequalities with p_i, $i = 1, \ldots, n$, variables and hence the feasibility of $p \geq\geq 0$ such that $z - Mp \geq\geq 0$ holds can be checked by standard linear matrix inequality (LMI) techniques (Boyd *et al.* 1994).

Remark 7: An identical theorem to Theorem 13 holds for lossless systems with linear supply rates $s(u, y) = q^{\mathrm{T}}y + r^{\mathrm{T}}u$. However, in this case (45) and (46) hold with $l = 0$ and $w = 0$.

7. Feedback interconnections of discrete-time non-negative dynamical systems

In this section we consider stability of feedback interconnections of non-negative dynamical systems. We begin by considering the non-linear non-negative dynamical system $\mathcal{G}$ given by (21) and (22) with the non-linear non-negative dynamical feedback system $\mathcal{G}_{\mathrm{c}}$ given by

$$x_{\mathrm{c}}(k+1) = f_{\mathrm{c}}(x_{\mathrm{c}}(k)) + G_{\mathrm{c}}(x_{\mathrm{c}}(k))u_{\mathrm{c}}(k), \quad x_{\mathrm{c}}(0) = x_{\mathrm{c}0}, \quad k \in \mathcal{N} \tag{47}$$

$$y_c(k) = h_{\mathrm{c}}(x_{\mathrm{c}}(k)) \tag{48}$$

where $f_{\mathrm{c}}: \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}}}$, $G_{\mathrm{c}}: \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}} \times m_{\mathrm{c}}}$, $h_{\mathrm{c}}: \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{l_{\mathrm{c}}}$, $f_{\mathrm{c}}(x_{\mathrm{c}}) \geq\geq 0$, $G_{\mathrm{c}}(x_{\mathrm{c}}) \geq\geq 0$ and $h_{\mathrm{c}}(x_{\mathrm{c}}) \geq\geq 0$, $x_{\mathrm{c}} \in \overline{\mathbb{R}}_+^{n_{\mathrm{c}}}$.

Theorem 14: *Let $q \in \mathbb{R}^l$, $r \in \mathbb{R}^m$, $q_{\mathrm{c}} \in \mathbb{R}^{l_{\mathrm{c}}}$, and $r_{\mathrm{c}} \in \mathbb{R}^{m_{\mathrm{c}}}$. Consider the non-linear non-negative dynamical systems $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ given by* (21), (22), *and* (47), (48), *respectively. Assume that $\mathcal{G}$ is dissipative with respect to the linear supply rate $s(u, y) = q^{\mathrm{T}}y + r^{\mathrm{T}}u$ and with a positive-definite storage function $V_{\mathrm{s}}(\cdot)$, and assume that $\mathcal{G}_{\mathrm{c}}$ is dissipative with respect to the linear supply rate $s_{\mathrm{c}}(u_{\mathrm{c}}, y_c) = q_{\mathrm{c}}^{\mathrm{T}}y_c + r_{\mathrm{c}}^{\mathrm{T}}u_{\mathrm{c}}$ and with a positive-definite storage function $V_{\mathrm{sc}}(\cdot)$. Then the following statements hold:*

(i) *If there exists a scalar $\sigma > 0$ such that $q + \sigma r_{\mathrm{c}} \leq\leq 0$ and $r + \sigma q_{\mathrm{c}} \leq\leq 0$, then the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is Lyapunov stable.*

(ii) *If $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are zero-state observable and there exists a scalar $\sigma > 0$ such that $q + \sigma r_{\mathrm{c}} << 0$ and $r + \sigma q_{\mathrm{c}} << 0$, then the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is asymptotically stable.*

(iii) *If $\mathcal{G}$ is zero-state observable, rank $G_{\mathrm{c}}(0) = m_{\mathrm{c}}$, $\mathcal{G}_{\mathrm{c}}$ is geometrically dissipative with respect to the supply rate $s_{\mathrm{c}}(u_{\mathrm{c}}, y_c) = q_{\mathrm{c}}^{\mathrm{T}}y_c + r_{\mathrm{c}}^{\mathrm{T}}u_{\mathrm{c}}$, and there exists a scalar $\sigma > 0$ such that $q + \sigma r_{\mathrm{c}} \leq\leq 0$ and $r + \sigma q_{\mathrm{c}} \leq\leq 0$, then the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is asymptotically stable.*

(iv) *If $\mathcal{G}$ is geometrically dissipative with respect to the supply rate $s(u, y) = q^{\mathrm{T}}y + r^{\mathrm{T}}u$, $\mathcal{G}_{\mathrm{c}}$ is geometrically dissipative with respect to the supply rate $s_{\mathrm{c}}(u_{\mathrm{c}}, y_c) = q_{\mathrm{c}}^{\mathrm{T}}y_c + r_{\mathrm{c}}^{\mathrm{T}}u_{\mathrm{c}}$, and there exists a scalar $\sigma > 0$ such that $q + \sigma r_{\mathrm{c}} \leq\leq 0$ and $r + \sigma q_{\mathrm{c}} \leq\leq 0$, then the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is asymptotically stable.*

Proof: Note that the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is given by $u = y_c$ and $u_{\mathrm{c}} = y$ so that the closed-loop dynamics of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is given by

$$\begin{bmatrix} x(k+1) \\ x_{\mathrm{c}}(k+1) \end{bmatrix} = \begin{bmatrix} f(x(k)) + G(x(k))h_{\mathrm{c}}(x_{\mathrm{c}}(k)) \\ f_{\mathrm{c}}(x_{\mathrm{c}}(k)) + G_{\mathrm{c}}(x_{\mathrm{c}}(k))h(x(k)) + G_{\mathrm{c}}(x_{\mathrm{c}}(k))J(x(k))h_{\mathrm{c}}(x_{\mathrm{c}}(k)) \end{bmatrix}$$

which implies that

$$\tilde{f}(\tilde{x}) \triangleq \begin{bmatrix} f(x) + G(x)h_{\mathrm{c}}(x_{\mathrm{c}}) \\ f_{\mathrm{c}}(x_{\mathrm{c}}) + G_{\mathrm{c}}(x_{\mathrm{c}})h(x) + G_{\mathrm{c}}(x_{\mathrm{c}})J(x)h_{\mathrm{c}}(x_{\mathrm{c}}) \end{bmatrix}$$

where $\tilde{x} = [x^{\mathrm{T}} \ x_c^{\mathrm{T}}]^{\mathrm{T}}$, is non-negative. Hence, the closed-loop system is also non-negative and thus $x(k) \geq\geq 0$, $x_{\mathrm{c}}(k) \geq\geq 0$, $u(k) \geq\geq 0$, $y(k) \geq\geq 0$, $k \in \mathcal{N}$. Now, the proof follows from Lyapunov theory and invariant set theorem arguments using the Lyapunov function candidate $V(x, x_{\mathrm{c}}) = V_{\mathrm{s}}(x) + \sigma V_{\mathrm{sc}}(x_{\mathrm{c}})$. □

Corollary 1: *Consider the non-linear non-negative dynamical systems $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ given by* (21), (22) *and* (47), (48), *respectively. Assume that $\mathcal{G}$ is non-accumulative with a positive-definite storage function $V_{\mathrm{s}}(\cdot)$, and assume that $\mathcal{G}_{\mathrm{c}}$ is geometrically non-accumulative with a positive-definite storage function $V_{\mathrm{sc}}(\cdot)$. Then the following statements hold:*

(i) *If $\mathcal{G}$ is zero-state observable and rank $G_{\mathrm{c}}(0) = m_{\mathrm{c}}$, then the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is asymptotically stable.*

(ii) *If $\mathcal{G}$ is geometrically non-accumulative, then the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is asymptotically stable.*

Proof: The result is a direct consequence of (iii) and (iv) of Theorem 14 with $\sigma = 1$, $q = -r_c = -e$ and $r = -q_c = e$. □

Next, we develop absolute stability criteria for discrete-time linear non-negative feedback systems with non-negative memoryless input non-linearities. Since absolute stability theory concerns the stability for classes of feedback non-linearities which, as noted in Haddad and Bernstein (1994), can readily be interpreted as an uncertainty model, the proposed framework can be used to analyse robustness of compartmental systems developed from data models. Specifically, given the discrete-time non-negative system $\mathcal{G}$ characterized by (43) and (44) we derive sufficient conditions that guarantee asymptotic stability of the feedback interconnection involving the discrete-time linear non-negative system $\mathcal{G}$ and the feedback non-negative input non-linearity $\sigma(\cdot,\cdot) \in \Phi$, where

$$\Phi \triangleq \Big\{ \sigma\colon \mathcal{N} \times \overline{\mathbb{R}}^l_+ \to \overline{\mathbb{R}}^m_+ \colon \sigma(\cdot, 0) = 0, \quad 0 \leq\leq \sigma(k, y) \leq\leq My, \ y \in \overline{\mathbb{R}}^l_+, \ k \in \mathcal{N} \Big\} \tag{49}$$

$M >> 0$ and $M \in \mathbb{R}^{m\times l}$.

Theorem 15: *Consider the non-negative dynamical system $\mathcal{G}$ given by* (43) *and* (44) *and assume that* (A, C) *is observable and $\mathcal{G}$ is geometrically dissipative with respect to the supply rate* $s(u, y) = e^{\mathrm{T}}u - e^{\mathrm{T}}My$, *where* $M >> 0$. *Then, the positive feedback interconnection of $\mathcal{G}$ and $\sigma(\cdot,\cdot)$ is globally asymptotically stable for all* $\sigma(\cdot,\cdot) \in \Phi$.

Proof: Since $\sigma(k, y) \geq\geq 0$ for all $k \in \mathcal{N}$, $y \in \overline{\mathbb{R}}^l_+$, and (43) and (44) is a discrete-time non-negative dynamical system, it follows that the positive feedback interconnection of $\mathcal{G}$ and $\sigma(\cdot,\cdot)$ given by

$$x(k+1) = Ax(k) + B\sigma(k, y(k)), \quad x(0) = x_0, \quad k \in \mathcal{N}$$

is a non-negative dynamical system for all $\sigma(\cdot,\cdot) \in \Phi$. Next, since (A, C) is observable and $\mathcal{G}$ is geometrically dissipative with respect to the supply rate $s(u, y) = e^{\mathrm{T}}u - e^{\mathrm{T}}My$, it follows from Remark 4 and Theorem 13 with $r = e$ and $q = -M^{\mathrm{T}}e$ that there exists $p \in \mathbb{R}^n_+$, $l \in \overline{\mathbb{R}}^n_+$ and $w \in \overline{\mathbb{R}}^m_+$, and a scalar $\rho > 1$ such that

$$0 = A^{\mathrm{T}}p - \frac{1}{\rho}p + C^{\mathrm{T}}M^{\mathrm{T}}e + l \tag{50}$$

$$0 = B^{\mathrm{T}}p + D^{\mathrm{T}}M^{\mathrm{T}}e - e + w \tag{51}$$

Next, consider the Lyapunov function candidate $V_s(x) = p^{\mathrm{T}}x$ and note that for $x \in \overline{\mathbb{R}}^n_+$, $x \neq 0$

$$\begin{aligned}
\Delta V_s(x) &= p^{\mathrm{T}}(Ax + B\sigma) - p^{\mathrm{T}}x + \frac{1}{\rho}p^{\mathrm{T}}x - \frac{1}{\rho}p^{\mathrm{T}}x \\
&= p^{\mathrm{T}}(Ax + B\sigma) - \frac{\rho - 1}{\rho}p^{\mathrm{T}}x - \frac{1}{\rho}p^{\mathrm{T}}x \\
&\leq p^{\mathrm{T}}(Ax + B\sigma) - \frac{1}{\rho}p^{\mathrm{T}}x \\
&= e^{\mathrm{T}}[\sigma - My] - l^{\mathrm{T}}x - w^{\mathrm{T}}\sigma \\
&\leq e^{\mathrm{T}}[\sigma - My]
\end{aligned}$$

Now, since $\sigma \leq\leq My$ for all $\sigma(\cdot,\cdot) \in \Phi$, it follows that $\Delta V_s(x) < 0$, $x \in \overline{\mathbb{R}}^n_+$, $x \neq 0$. Hence, the positive feedback interconnection of $\mathcal{G}$ and $\sigma(\cdot,\cdot)$ is globally asymptotically stable for all $\sigma(\cdot,\cdot) \in \Phi$. □

Remark 8: To consider non-linearities with upper and lower bounds of the form $M_1 y \leq\leq \sigma(k, y) \leq\leq M_2 y$, where $\sigma(\cdot,\cdot) \in \Phi$, we can use the standard loop shifting techniques discussed in Khalil (1996, p. 408). In this case, Theorem 15 holds with $\sigma(k, y)$, A, B, C, D and M replaced by $\sigma(k, y) - M_1 y$, $A + B(I - M_1 D)^{-1} \times M_1 C$, $B(I - M_1 D)^{-1}$, $(I - DM_1)^{-1}C$, $(I - DM_1)^{-1}D$ and $M_2 - M_1$, respectively.

Theorem 16: *Consider the non-linear non-negative dynamical system $\mathcal{G}$ given by* (21) *and* (22), *where* $f\colon \mathcal{D} \to \mathbb{R}^n$ *is non-negative,* $G(x) \geq\geq 0$, $h(x) \geq\geq 0$ *and* $J(x) \geq\geq 0$, $x \in \overline{\mathbb{R}}^n_+$. *Suppose there exist functions* $V_s\colon \overline{\mathbb{R}}^n_+ \to \overline{\mathbb{R}}_+$ *and* $P_{1u}\colon \overline{\mathbb{R}}^n_+ \to \overline{\mathbb{R}}^{1\times m}_+$, *such that* $V_s(\cdot)$ *is positive definite,* $V_s(0) = 0$, $V_s(x) \to \infty$ *as* $\|x\| \to \infty$, *and*

$$0 = V_s(f(x)) - \frac{1}{\rho}V_s(x) - e^{\mathrm{T}}Mh(x) + \ell(x) \tag{52}$$

$$0 = P_{1u}(x) - e^{\mathrm{T}}MJ(x) - e^{\mathrm{T}} + \mathcal{W}^{\mathrm{T}}(x) \tag{53}$$

Then, the positive feedback interconnection of $\mathcal{G}$ and $\sigma(\cdot,\cdot)$ is globally asymptotically stable for all $\sigma(\cdot,\cdot) \in \Phi$.

Proof: The proof is similar to that of Theorem 15 and hence is omitted. □

8. Illustrative example

In this section, we provide an example to demonstrate the utility of some of the basic mathematical results developed in the paper. This example models the flow of thyroxine when injected into the blood stream and then carried into the liver where it is converted into iodine which in turn is absorbed into the bile. A simple yet accurate model for the flow of thyroxine into the blood stream is captured by the three-compartment model shown in figure 2.

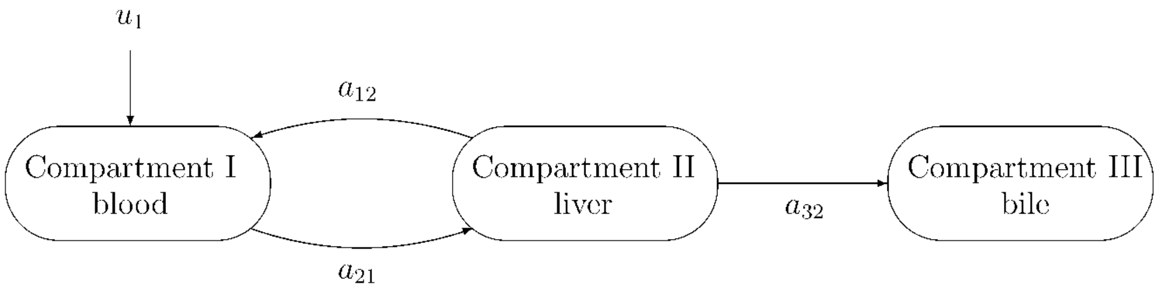

Figure 2. Three-compartment thyroxine model.

The first compartment corresponds to the blood plasma, the second compartment to the liver, and the third compartment to the bile. Note that the feedback path between Compartments I and II reflects the fact that neither conversion nor absorption occurs instantaneously; that is, a fraction of the thyroxine that enters the liver is fed back to the blood before conversion to iodine. To analyse this compartmental system let, for $i, j = 1, 2, 3$, x_i denote the concentration of thyroxine in compartment i, a_{ij} denote the instantaneous transfer coefficient of thyroxine flow rates from compartment j to compartment i in units of $(\text{time})^{-1}$, and u_1 denote a bolus (impulse) injection into the blood stream. Since at $t = 0$ no absorption or transfer of thyroxine can occur it follows that $x_2(0) = x_3(0) = 0$. Furthermore, since the input material is a bolus injection we can always reproduce the impulsive response with the free response by setting $x_3(0) = bv$, where $v \in \mathbb{R}$ denotes the impulse strength. Hence, a mass balance of the three-state compartment model shown in figure 2 yields (3) with

$$A_c = \begin{bmatrix} -a_{21} & a_{12} & 0 \\ a_{21} & -(a_{12} + a_{32}) & 0 \\ 0 & a_{32} & 0 \end{bmatrix} \tag{54}$$

Now, it follows that the discretization of (3) is given by (4) where $A_d = e^{A_c h}$ and h is the sampling rate. Note that since A_c is essentially non-negative it follows from Lemma 2.2 of Haddad *et al.* (2001) that $A_d \geq\geq 0$. Hence, $A_d^k \geq\geq 0$, $k \in \mathcal{N}$, and consequently if $x(0)$ is non-negative, then the solution $x(h) = A_d^k x(0)$ is non-negative for all $k \in \mathcal{N}$.

Next, we assume that the sample rate h is small so that $A_d = e^{A_c h} = \sum_{i=0}^{\infty} (i!)^{-1} (A_c h)^i \approx I + hA_c$. In this case

$$A_d = \begin{bmatrix} 1 - ha_{21} & ha_{12} & 0 \\ ha_{21} & 1 - h(a_{12} + a_{32}) & 0 \\ 0 & ha_{32} & 1 \end{bmatrix} \tag{55}$$

Furthermore, since $A_d - I$ is singular it follows that the set of equilibria $\mathcal{E} = \{(x_1, x_2, x_3) \in \overline{\mathbb{R}}_+^3 : A_d x = x\} = \mathcal{N}(A_d - I) = \{(x_1, x_2, x_3) \in \overline{\mathbb{R}}_+^3 : (0, 0, x_3)\}$. Now, taking $p = e >> 0$ and $r = [0 \ 0 \ 0]^T$, it follows that (2) holds and hence the discretized system is Lyapunov stable by (i) of Theorem 1. In addition, since

$$\begin{aligned} \operatorname{spec}(A_d) &= \Big\{ 1 - \frac{h}{2}(a_{12} + a_{21} + a_{32}) - \frac{h}{2}\sqrt{(a_{12} + a_{21} + a_{32})^2 - 4a_{21}a_{32}}, \\ &\quad 1 - \frac{h}{2}(a_{12} + a_{21} + a_{32}) - \frac{h}{2}\sqrt{(a_{12} + a_{21} + a_{32})^2 - 4a_{21}a_{32}}, 1 \Big\} \end{aligned}$$

it follows from Theorem 2 that the discretized system is semistable. Finally, we show that the discretized system is non-accumulative. Here, we assume $u_1(k)$ is an arbitrary discrete input and the bile is discharged into the duodenum so that $y(k) = Cx(k)$, where $C = [0, 0, a_{33}]$ and $a_{33} > 0$. In this case, using the storage function $V_s(x_1, x_2, x_3) = x_1 + x_2 + x_3$ it follows that $\Delta V_s(x_1, x_2, x_3) \leq u_1 - y$ and hence the discretized system is non-accumulative.

9. Conclusion

In this paper we developed stability results for discrete-time linear and non-linear non-negative and compartmental systems using linear Lyapunov functions. In addition, dissipativity results for discrete-time non-linear non-negative systems with linear and non-linear storage functions and linear supply rates were also developed. Furthermore, we developed *new* Kalman–Yakubovich–Popov conditions in terms of the non-negative system dynamics for characterizing dissipativeness via system storage functions and linear supply rates for non-negative dynamical systems. Finally, general stability criteria were given for Lyapunov and asymptotic stability of feedback interconnections of discrete-time linear and non-linear non-negative dynamical systems.

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