INT. J. CONTROL, 15 FEBRUARY 2004, VOL. 77, NO. 3, 250–263

A Lyapunov-based adaptive control framework for discrete-time non-linear systems with exogenous disturbances

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A direct adaptive non-linear control framework for discrete-time multivariable non-linear uncertain systems with exogenous bounded disturbances is developed. The adaptive non-linear controller addresses adaptive stabilization, disturbance rejection and adaptive tracking. The proposed framework is Lyapunov-based and guarantees partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. In the case of bounded energy ℓ_2 disturbances the proposed approach guarantees a non-expansivity constraint on the closed-loop input–output map. Finally, three illustrative numerical examples are provided to demonstrate the efficacy of the proposed approach.

1. Introduction

The purpose of feedback control is to achieve desirable system performance in the face of system uncertainty and system disturbances. Although system identification can reduce uncertainty to some extent, residual modelling discrepancies always remain. Controllers must therefore be robust to achieve desired disturbance rejection and/or tracking performance requirements in the presence of such modelling uncertainty. To this end, adaptive control along with robust control theory have been developed to address the problem of system performance in the face of system uncertainty in control-system design without excessive reliance on system models.

Adaptive controllers directly or indirectly adjust feedback gains to maintain closed-loop stability and improve performance in the face of system errors. Specifically, indirect adaptive controllers utilize parameter update laws to estimate unknown system parameters and adjust feedback gains to account for system variation, while direct adaptive controllers directly adapt the controller gains in response to system variations. Even though adaptive control algorithms have been developed in the literature for both continuous-time and discrete-time systems, the majority of the discrete-time results are based on recursive least squares and least mean squares algorithms (Egardt 1980, Fuchs 1980, Goodwin and Long 1980, Goodwin *et al.* 1980, Narendra and Lin 1980) with primary focus on state convergence. Alternatively, Lyapunov-based adaptive controllers have been developed for continuous-time systems guaranteeing asymptotic stability of the system states (see, e.g. Narendra and Annaswamy 1989, Krstić *et al.* 1995, Kaufman *et al.* 1998). Notable Lyapunov-based adaptive control algorithms for discrete-time systems are given in Johansson (1989), Yeh and Kokotović (1995), Rokui and Khorasani (1997) and Venugopal *et al.* (2003). However, the literature on discrete-time adaptive disturbance rejection control using Lyapunov methods is virtually non-existent.

For discrete-time dynamical systems, Lyapunov-based frameworks for adaptive control are quite intricate since the Lyapunov difference does not remove terms involving the model reference stabilizing gain from the resulting Lyapunov difference expression. This leads to asymptotic non-positivity of the Lyapunov difference and thus Lyapunov stability cannot be guaranteed (Venugopal *et al.* 2003). This difficulty was first pointed out by Kanellakopoulos (1994) and is the main reason why Lyapunov-based discrete-time adaptive control is *not* a straightforward extension of continuous-time adaptive control theory. As a result, most of the discrete-time adaptive model reference and tracking control results are based on the classical key technical lemma which does not guarantee Lyapunov stability.

In this paper, using a logarithmic Lyapunov function we develop a Lyapunov-based direct adaptive control framework for adaptive stabilization, disturbance rejection and command following of multivariable discrete-time non-linear uncertain systems with exogenous bounded amplitude disturbances and ℓ_2 disturbances. These results are analogous to, but by no means a direct extension of, the recent continuous-time adaptive disturbance rejection results in Haddad and Hayakawa (2002) for continuous-time non-linear uncertain systems. In contrast to the results presented in Haddad and Hayakawa (2002), logarithmic Lyapunov functions are shown to be essential for discrete-time Lyapunov-based adaptive control. Specifically, a logarithmic Lyapunov-based direct adaptive control framework is developed that guarantees partial asymptotic stability

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Received 1 August 2002. Revised 19 November 2003.

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International Journal of Control ISSN 0020–7179 print/ISSN 1366–5820 online © 2004 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/00207170310001649900

of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Furthermore, in the case where the non-linear system is represented in normal form, the non-linear discrete-time adaptive controller is constructed *without* requiring knowledge of the system dynamics or system disturbances. In the case where the system disturbances are ℓ_2 disturbances, the proposed framework guarantees that the closed-loop non-linear input–output map from uncertain exogenous ℓ_2 disturbances to system performance variables is non-expansive and the solution of the closed-loop system is partially asymptotically stable. The proposed adaptive controller thus addresses the problem of disturbance rejection for non-linear uncertain discrete-time systems with bounded energy (square-summable) ℓ_2 signal norms on the disturbances and performance variables.

The contents of the paper are as follows. In § 2 we present our main direct adaptive control framework for adaptive stabilization, disturbance rejection, and command following of multivariable non-linear uncertain discrete-time systems with matched exogenous bounded disturbances. In § 3 we extend the results of § 2 to non-linear uncertain discrete-time systems with exogenous ℓ_2 disturbances without a matching condition requirement. Three illustrative numerical examples are presented in § 4 to demonstrate the efficacy of the proposed direct adaptive stabilization and tracking framework. Finally, in § 5 we draw some conclusions.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $(\ )^{\mathrm{T}}$ denotes transpose, $(\ )^{\dagger}$ denotes the Moore–Penrose generalized inverse, and $\mathcal{N}$ denotes the set of non-negative integers. Furthermore, we write $\lambda_{\min}(M)$ (resp. $\lambda_{\max}(M)$) for the minimum (resp. maximum) eigenvalue of the Hermitian matrix M, $\sigma_{\max}(M)$, for the maximum singular value of the matrix M, $\operatorname{tr}(\cdot)$ for the trace operator, and $\ln(\cdot)$ for the natural log operator.

2. Discrete-time adaptive control for non-linear systems with exogenous disturbances

In this section we consider the problem of characterizing adaptive feedback control laws for non-linear uncertain discrete-time systems with exogenous disturbances. Specifically, consider the controlled non-linear uncertain discrete-time system $\mathcal{G}$ given by

$$x(k+1) = f(x(k)) + G(x(k))u(k) + J(x(k))w(k), \quad x(0) = x_0, \quad k \in \mathcal{N} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $k \in \mathcal{N}$, is the state vector, $u(k) \in \mathbb{R}^m$, $k \in \mathcal{N}$, is the control input, $w(k) \in \mathbb{R}^d$, $k \in \mathcal{N}$, is a known bounded disturbance vector such that $\|w(k)\|_2 \le \delta$, $k \in \mathcal{N}$, $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(0) = 0$, $G\colon \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is such that $\operatorname{rank} G(x) = m$, $x \in \mathbb{R}^n$, and $J\colon \mathbb{R}^n \to \mathbb{R}^{n\times d}$ is a disturbance weighting matrix function with *unknown* entries. Note that even though $w(k)$, $k \in \mathcal{N}$, is assumed to be known, the disturbance signal $J(x(k))w(k)$, $k \in \mathcal{N}$, is an *unknown* bounded disturbance. The control input $u(\cdot)$ in (1) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(k) \in \mathbb{R}^m, k \in \mathcal{N}$.

Theorem 1: *Consider the non-linear system $\mathcal{G}$ given by* (1). *Assume there exist a matrix $K_{\mathrm{g}} \in \mathbb{R}^{m\times s}$, functions $V_{\mathrm{s}}\colon \mathbb{R}^n \to \mathbb{R}$, $\hat{G}\colon \mathbb{R}^n \to \mathbb{R}^{m\times m}$, $F\colon \mathbb{R}^n \to \mathbb{R}^s$, $P_{1u}\colon \mathbb{R}^n \to \mathbb{R}^{1\times m}$, $\ell\colon \mathbb{R}^n \to \mathbb{R}^t$, a non-negative-definite matrix function $P_{2u}\colon \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and positive constants $\bar{\gamma}$, ε, μ and ν such that $V_{\mathrm{s}}(\cdot)$ and $\ell(\cdot)$ are continuous, $V_{\mathrm{s}}(0) = 0$, $\ell(0) = 0$, $\det \hat{G}(x) \ne 0$, $x \in \mathbb{R}^n$, $\hat{G}^{\mathrm{T}}(x)P_{2u}(x)\hat{G}(x) \le \nu I_m$, $x \in \mathbb{R}^n$, and for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$*

$$V_{\mathrm{s}}(f(x) + G(x)u) = V_{\mathrm{s}}(f(x)) + P_{1u}(x)u + u^{\mathrm{T}}P_{2u}(x)u \tag{2}$$

$$0 \ge V_{\mathrm{s}}(f_{\mathrm{s}}(x)) - V_{\mathrm{s}}(x) + \ell^{\mathrm{T}}(x)\ell(x) + \varepsilon P_{1u}(x)\hat{G}(x)\hat{G}^{\mathrm{T}}(x)P_{1u}^{\mathrm{T}}(x) \tag{3}$$

$$F^{\mathrm{T}}(x)F(x) \le \bar{\gamma}x^{\mathrm{T}}x, \qquad x \in \mathbb{R}^n \tag{4}$$

$$V_{\mathrm{s}}(x) \ge \mu x^{\mathrm{T}}x \tag{5}$$

where

$$f_{\mathrm{s}}(x) \triangleq f(x) + G(x)\hat{G}(x)K_{\mathrm{g}}F(x) \tag{6}$$

Furthermore, assume there exists a matrix $\Psi \in \mathbb{R}^{m\times d}$ such that $G(x)\hat{G}(x)\Psi = J(x)$. Finally, let $\tilde{x}(k) \triangleq [F^{\mathrm{T}}(x(k)), w^{\mathrm{T}}(k)]^{\mathrm{T}}$, $c > 0$, and $Q \in \mathbb{R}^{m\times m}$ be positive definite such that $\lambda_{\max}(Q) < 2$. Then the adaptive feedback control law

$$u(k) = \hat{G}(x(k))K(k)\tilde{x}(k) \tag{7}$$

where $K(k) \in \mathbb{R}^{m\times(s+d)}$, $k \in \mathcal{N}$, with update law

$$K(k+1) = K(k) - \frac{1}{c + \tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}Q\hat{G}^{-1}(x(k)) \times G^{\dagger}(x(k))[x(k+1) - f_{\mathrm{s}}(x(k))]\tilde{x}^{\mathrm{T}}(k) \tag{8}$$

guarantees that the solution $(x(k), K(k)) \equiv (0, [K_{\mathrm{g}}, -\Psi])$ of the closed-loop system given by (1), (7) *and* (8) *is Lyapunov stable and $\ell(x(k)) \to 0$ as $k \to \infty$. If, in addition, $\ell^{\mathrm{T}}(x)\ell(x) > 0$, $x \in \mathbb{R}^n$, $x \ne 0$, then $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$.*

Proof: First, define $\tilde{K}(k) \triangleq K(k) - \hat{K}_{\mathrm{g}}$ and $\tilde{u}(k) \triangleq \tilde{K}(k)\tilde{x}(k)$, where $\hat{K}_{\mathrm{g}} \triangleq [K_{\mathrm{g}}, -\Psi]$. Note that with $u(k)$, $k \in \mathcal{N}$, given by (7) it follows from (1) that

$$x(k+1) = f(x(k)) + G(x(k))\hat{G}(x(k))K(k)\tilde{x}(k) + J(x(k))w(k), \qquad x(0) = x_0, \quad k \in \mathcal{N} \tag{9}$$

or, equivalently, using (6) and the fact that $G(x)\hat{G}(x)\Psi = J(x)$

$$\begin{aligned} x(k+1) &= f_{\mathrm{s}}(x(k)) + G(x(k))\hat{G}(x(k))\tilde{K}(k)\tilde{x}(k) \\ &= f_{\mathrm{s}}(x(k)) + G(x(k))\hat{G}(x(k))\tilde{u}(k), \\ &\qquad x(0) = x_0, \quad k \in \mathcal{N} \end{aligned} \tag{10}$$

Furthermore, note that adding and subtracting $\hat{K}_{\mathrm{g}}$ to and from (8) and using (10) it follows that

$$\begin{aligned} \tilde{K}(k+1) &= \tilde{K}(k) - \frac{1}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)} Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k)) \\ &\qquad \times \Big[G(x(k))\hat{G}(x(k))\tilde{K}(k)\tilde{x}(k)\Big]\tilde{x}^{\mathrm{T}}(k) \\ &= \tilde{K}(k) - \frac{1}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)} Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k), \qquad k \in \mathcal{N} \end{aligned} \tag{11}$$

To show Lyapunov stability of the closed-loop system (10) and (11), consider the Lyapunov function candidate

$$V(x,K) = \ln(1 + V_{\mathrm{s}}(x)) + a\,\mathrm{tr}(K - \hat{K}_{\mathrm{g}})^{\mathrm{T}} Q^{-1}(K - \hat{K}_{\mathrm{g}}) \tag{12}$$

where

$$a \geq \frac{(1/4\varepsilon) + \nu}{\lambda_{\min}(2I - Q)} \max\left\{\delta^2 + c, \frac{\bar{\gamma}}{\mu}\right\} \tag{13}$$

Note that $V(0, \hat{K}_{\mathrm{g}}) = 0$ and, since $V_{\mathrm{s}}(\cdot)$ and Q are positive definite and $a > 0$, $V(x,K) > 0$ for all $(x,K) \neq (0, \hat{K}_{\mathrm{g}})$. Furthermore, $V(x,K)$ is radially unbounded. Now, letting $x(k)$, $k \in \mathcal{N}$, denote the solution to (10) and using (2), (3) and (11), it follows that the Lyapunov difference along the closed-loop system trajectories is given by

$$\begin{aligned} &\Delta V(x(k), K(k)) \\ &\quad \triangleq V(x(k+1), K(k+1)) - V(x(k), K(k)) \\ &\quad = \ln\Big(1 + V_{\mathrm{s}}(f_{\mathrm{s}}(x(k)) + G(x(k))\hat{G}(x(k))\tilde{u}(k))\Big) \\ &\qquad + a\,\mathrm{tr}\left(\tilde{K}(k) - \frac{1}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)} Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k)\right)^{\mathrm{T}} Q^{-1} \\ &\qquad \times \left(\tilde{K}(k) - \frac{1}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)} Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k)\right) \\ &\qquad - \ln(1 + V_{\mathrm{s}}(x(k))) - a\,\mathrm{tr}\tilde{K}^{\mathrm{T}}(k)Q^{-1}\tilde{K}(k) \\ &\quad = \ln\Big(1 + \Big[V_{\mathrm{s}}(f_{\mathrm{s}}(x(k))) + P_{1u}(x(k))\hat{G}(x(k))\tilde{u}(k) \\ &\qquad + \tilde{u}^{\mathrm{T}}(k)\hat{G}^{\mathrm{T}}(x(k))P_{2u}(x(k))\hat{G}(x(k))\tilde{u}(k) - V_{\mathrm{s}}(x(k))\Big] \\ &\qquad \times [1 + V_{\mathrm{s}}(x(k))]^{-1}\Big) + a\,\mathrm{tr}\tilde{K}^{\mathrm{T}}(k)Q^{-1}\tilde{K}(k) \\ &\qquad - \frac{2a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\mathrm{tr}\tilde{K}^{\mathrm{T}}(k)\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k) \\ &\qquad + \frac{a}{(c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k))^2}\mathrm{tr}\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k) \\ &\qquad - a\,\mathrm{tr}\tilde{K}^{\mathrm{T}}(k)Q^{-1}\tilde{K}(k) \\ &\quad \leq \Big[-\ell^{\mathrm{T}}(x(k))\ell(x(k)) - \varepsilon P_{1u}(x(k))\hat{G}(x(k))\hat{G}^{\mathrm{T}}(x(k)) \\ &\qquad \times P_{1u}^{\mathrm{T}}(x(k)) + P_{1u}(x(k))\hat{G}(x(k))\tilde{u}(k) + \nu\tilde{u}^{\mathrm{T}}(k)\tilde{u}(k)\Big] \\ &\qquad \times [1 + V_{\mathrm{s}}(x(k))]^{-1} - \frac{2a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)\tilde{K}(k)\tilde{x}(k) \\ &\qquad + \frac{a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)Q\tilde{K}(k)\tilde{x}(k) \end{aligned} \tag{14}$$

where in (14) we used $\ln a - \ln b = \ln(a/b)$ and $\ln(1+c) \leq c$ for $a, b > 0$ and $c \geq -1$, respectively, and $\tilde{x}^{\mathrm{T}}\tilde{x}/(c + \tilde{x}^{\mathrm{T}}\tilde{x}) < 1$. Now, adding and subtracting

$$\frac{1}{4\varepsilon}\frac{\tilde{u}^{\mathrm{T}}(k)\tilde{u}(k)}{1 + V_{\mathrm{s}}(x(k))}$$

to and from (14) and collecting terms yields

$$\begin{aligned} &\Delta V(x(k), K(k)) \\ &\quad \leq -\frac{1}{1 + V_{\mathrm{s}}(x(k))}\ell^{\mathrm{T}}(x(k))\ell(x(k)) \\ &\qquad - \frac{1}{1 + V_{\mathrm{s}}(x(k))}\big[P_{1u}(x(k)),\ \tilde{u}^{\mathrm{T}}(k)\big] \\ &\qquad \times \begin{bmatrix} \varepsilon\hat{G}(x(k))\hat{G}^{\mathrm{T}}(x(k)) & -\frac{1}{2}\hat{G}(x(k)) \\ -\frac{1}{2}\hat{G}^{\mathrm{T}}(x(k)) & (1/4\varepsilon)I_m \end{bmatrix} \begin{bmatrix} P_{1u}^{\mathrm{T}}(x(k)) \\ \tilde{u}(k) \end{bmatrix} \\ &\qquad + \frac{1}{1 + V_{\mathrm{s}}(x(k))}\left[\frac{1}{4\varepsilon}\tilde{u}^{\mathrm{T}}(k)\tilde{u}(k) + \nu\tilde{u}^{\mathrm{T}}(k)\tilde{u}(k)\right] \\ &\qquad - \frac{a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)(2I_m - Q)\tilde{K}(k)\tilde{x}(k) \\ &\quad \leq -\frac{\ell^{\mathrm{T}}(x(k))\ell(x(k))}{1 + V_{\mathrm{s}}(x(k))} \\ &\qquad - \frac{\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)\tilde{R}(x(k), w(k))\tilde{K}(k)\tilde{x}(k)}{(c + \tilde{x}^{\mathrm{T}}(k)\tilde{x}(k))(1 + V_{\mathrm{s}}(x(k)))}, \qquad k \in \mathcal{N} \end{aligned} \tag{15}$$

where

$$\tilde{R}(x,w) \triangleq a(1 + V_{\mathrm{s}}(x))(2I_m - Q) - \left(\frac{1}{4\varepsilon} + \nu\right)(c + \tilde{x}^{\mathrm{T}}\tilde{x})I_m \tag{16}$$

Noting that $2I_m - Q > 0$, since by assumption $\lambda_{\max}(Q) < 2$, and a satisfies (13), it follows that

$$\begin{aligned}\tilde{R}(x, w) &\geq a(1 + \mu x^{\mathrm{T}} x)(2I_m - Q) \\ &\quad - \left(\frac{1}{4\varepsilon} + \nu\right)(\delta^2 + c + F^{\mathrm{T}}(x)F(x))I_m \\ &\geq a(1 + \mu x^{\mathrm{T}} x)(2I_m - Q) \\ &\quad - \left(\frac{1}{4\varepsilon} + \nu\right)(\delta^2 + c + \bar{\gamma} x^{\mathrm{T}} x)I_m \\ &\geq 0, \qquad (x, w) \in \mathbb{R}^n \times \mathbb{R}^d \end{aligned} \tag{17}$$

Hence, the Lyapunov difference given by (15) yields

$$\begin{aligned}\Delta V(x(k), K(k)) &\leq -\frac{\ell^{\mathrm{T}}(x(k))\ell(x(k))}{1 + V_{\mathrm{s}}(x(k))} \\ &\quad - \frac{\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)\tilde{R}(x(k), w(k))\tilde{K}(k)\tilde{x}(k)}{\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)(1 + V_{\mathrm{s}}(x(k)))} \\ &\leq 0, \qquad k \in \mathcal{N} \end{aligned} \tag{18}$$

which proves that the solution $(x(k), K(k)) \equiv (0, \hat{K}_{\mathrm{g}})$ to (10) and (11) is Lyapunov stable. Furthermore, it follows from (the discrete-time version of Theorem 2 of Chellaboina and Haddad (2002) that $\ell(x(k)) \to 0$ as $k \to \infty$. Finally, if $\ell^{\mathrm{T}}(x)\ell(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$. □

Remark 1: Note that in the case where $\ell^{\mathrm{T}}(x)\ell(x) > 0$, $x \in \mathbb{R}^n, x \neq 0$, the conditions in Theorem 1 imply that $x(k) \to 0$ as $k \to \infty$ and hence it follows from (8) that $(x(k), K(k)) \to \mathcal{M} \triangleq \{(x, K) \in \mathbb{R}^n \times \mathbb{R}^{m \times (s+d)}: x = 0$ and $K(k+1) = K(k)\}$ as $k \to \infty$.

Remark 2: Theorem 1 is also valid for *time-varying* uncertain dynamical systems $\mathcal{G}_{\mathrm{k}}$ of the form

$$x(k+1) = f(k, x(k)) + G(k, x(k))u(k) + J(k, x(k))w(k), \qquad x(0) = x_0, \quad k \in \mathcal{N} \tag{19}$$

where $f: \mathcal{N} \times \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(k, 0) = 0, k \in \mathcal{N}$, $G: \mathcal{N} \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ and $J: \mathcal{N} \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$. In particular, replacing $F: \mathbb{R}^n \to \mathbb{R}^s$ by $F: \mathcal{N} \times \mathbb{R}^n \to \mathbb{R}^s$ and $\hat{G}: \mathbb{R}^n \to \mathbb{R}^{m \times m}$ by $\hat{G}: \mathcal{N} \times \mathbb{R}^n \to \mathbb{R}^{m \times m}$, and requiring $F^{\mathrm{T}}(k, x)F(k, x) \leq \bar{\gamma} x^{\mathrm{T}} x$, $\bar{\gamma} > 0$, $k \in \mathcal{N}$, in place of (4) and $G(k, x)\hat{G}(k, x)\Psi = J(k, x)$ in place of $G(x)\hat{G}(x)\Psi = J(x)$, it follows by using identical arguments as in the proof of Theorem 1 that the adaptive feedback control law

$$u(k) = \hat{G}(k, x(k))K(k)\tilde{x}(k) \tag{20}$$

where $\tilde{x}(k) \triangleq [F^{\mathrm{T}}(k, x(k)), w^{\mathrm{T}}(k)]^{\mathrm{T}}$, with update law

$$\begin{aligned}K(k+1) = K(k) - \frac{1}{c + \tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)} Q\hat{G}^{-1}(k, x(k))G^{\dagger}(k, x(k)) \\ \times [x(k+1) - f_{\mathrm{s}}(x(k))]\tilde{x}^{\mathrm{T}}(k) \end{aligned} \tag{21}$$

where $f_{\mathrm{s}}(x) = f(k, x) + G(k, x)\hat{G}(k, x)K_{\mathrm{g}}F(k, x)$, guarantees that the solution $(x(k), K(k)) \equiv (0, [K_{\mathrm{g}}, -\Psi])$ of the closed-loop system (19)–(21) is Lyapunov stable and $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$.

Remark 3: It follows from Remark 2 that Theorem 1 can also be used to construct adaptive tracking controllers for non-linear uncertain dynamical systems. Specifically, let $r_{\mathrm{d}}(k) \in \mathbb{R}^n$, $k \in \mathcal{N}$, denote a command input and define the error state $e(k) \triangleq x(k) - r_{\mathrm{d}}(k)$. In this case, the error dynamics are given by

$$\begin{aligned}e(k+1) &= f_{\mathrm{k}}(k, e(k)) + G_{\mathrm{k}}(k, e(k))u(k) \\ &\quad + J_{\mathrm{k}}(k, e(k))w_{\mathrm{k}}(k), \quad e(0) = e_0, \quad k \in \mathcal{N} \end{aligned} \tag{22}$$

where $f_{\mathrm{k}}(k, e(k)) = f(e(k) + r_{\mathrm{d}}(k)) - n(k)$ with $f(r_{\mathrm{d}}(k)) = n(k)$, $G_{\mathrm{k}}(k, e(k)) = G(e(k) + r_{\mathrm{d}}(k))$ and $J_{\mathrm{k}}(k, e(k))w_{\mathrm{k}}(k) = n(k) - r_{\mathrm{d}}(k+1) + J(e(k) + r_{\mathrm{d}}(k))w(k)$. Now, the adaptive tracking control law (20) and (21), with $x(k)$ replaced by $e(k)$, guarantees that $e(k) \to 0$ as $k \to \infty$ for all $e_0 \in \mathbb{R}^n$.

It is important to note that the adaptive control law (7) and (8) does *not* require explicit knowledge of the gain matrix K_{g}, the disturbance matching matrix Ψ, the disturbance weighting matrix function $J(x)$, and the positive constants ν, $\bar{\gamma}$, ε and μ; even though Theorem 1 requires the existence of K_{g} and Ψ along with the construction of $F(x)$, $\hat{G}(x)$ and $V_{\mathrm{s}}(x)$ such that $G(x)\hat{G}(x)\Psi = J(x)$ and (2)–(5) hold. Furthermore, if (1) is in normal form with asymptotically stable internal dynamics (Isidori 1995) and if the linear growth condition $f^{\mathrm{T}}(x)f(x) \leq \hat{\gamma} x^{\mathrm{T}} x$, $x \in \mathbb{R}^n$, $\hat{\gamma} > 0$, holds, then we can always construct functions $V_{\mathrm{s}}: \mathbb{R}^n \to \mathbb{R}$, $F: \mathbb{R}^n \to \mathbb{R}^s$, and $\hat{G}: \mathbb{R}^n \to \mathbb{R}^{n \times m}$ such that (2)–(5) hold *without* requiring knowledge of the system dynamics. For simplicity of exposition in the ensuing discussion we assume that $J(x) = D$, where $D \in \mathbb{R}^{n \times d}$ is a disturbance weighting matrix with unknown entries.

To elucidate the above discussion assume that the non-linear uncertain system $\mathcal{G}$ is generated by the difference model

$$z_i(k + \tau_i) = f_{\mathrm{u}i}(z(k)) + \sum_{j=1}^{m} G_{\mathrm{s}(i,j)}(z(k))u_j(k) + \sum_{l=1}^{d} \hat{D}_{(i,l)}w_l(k), \qquad k \in \mathcal{N}, \ i = 1, \ldots, m \tag{23}$$

where $\tau_i \in \mathcal{N}$ denotes the time delay (or relative degree) with respect to the output $z_i, z(k) = [z_1(k), \ldots, z_1(k + \tau_1 - 1), \ldots, z_m(k), \ldots, z_m(k + \tau_m - 1)]$, $z(0) = z_0$, $\hat{D}_{(i,l)} \in \mathbb{R}$, $i = 1, \ldots, m$, $l = 1, \ldots, d$, and $w_l(k) \in \mathbb{R}$, $k \in \mathcal{N}$, $l = 1, \ldots, d$. Here, we assume that the square matrix function $G_{\mathrm{s}}(z)$ composed of the entries $G_{\mathrm{s}(i,j)}(z)$, $i, j = 1, \ldots, m$, is such that $\det G_{\mathrm{s}}(z) \neq 0$, $z \in \mathbb{R}^{\hat{\tau}}$, where $\hat{\tau} = \tau_1 + \cdots + \tau_m$. Furthermore, since (23) is in a form where it does not possess internal dynamics, it follows that $\hat{\tau} = n$. The case where (23) possesses input-to-state stable internal dynamics can be analogously handled as shown in Haddad and Hayakawa (2002).

Next, define $x_i(k) \overset{\triangle}{=} [z_i(k), \ldots, z_i(k + \tau_i - 2)]^{\mathrm{T}}$, $i = 1, \ldots, m$, $x_{m+1}(k) \overset{\triangle}{=} [z_1(k + \tau_1 - 1), \ldots, z_m(k + \tau_m - 1)]^{\mathrm{T}}$ and $x(k) \overset{\triangle}{=} [x_1^{\mathrm{T}}(k), \ldots, x_{m+1}^{\mathrm{T}}(k)]^{\mathrm{T}}$ so that (23) can be described by (1) with

$$f(x) = \tilde{A}x + \tilde{f}_{\mathrm{u}}(x), \qquad G(x) = \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(x) \end{bmatrix},$$

$$J(x) = D = \begin{bmatrix} 0_{(n-m)\times d} \\ \hat{D} \end{bmatrix} \tag{24}$$

where

$$\tilde{A} = \begin{bmatrix} A_0 \\ 0_{m\times n} \end{bmatrix}, \qquad \tilde{f}_{\mathrm{u}}(x) = \begin{bmatrix} 0_{(n-m)\times 1} \\ f_{\mathrm{u}}(x) \end{bmatrix}$$

$A_0 \in \mathbb{R}^{(n-m)\times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation (Chen 1984), $f_{\mathrm{u}}: \mathbb{R}^n \to \mathbb{R}^m$ is an unknown function and satisfies $f_{\mathrm{u}}^{\mathrm{T}}(x) f_{\mathrm{u}}(x) \le \gamma_{\mathrm{u}} x^{\mathrm{T}} x$, $x \in \mathbb{R}^n$, where $\gamma_{\mathrm{u}} > 0$, $G_{\mathrm{s}}: \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and $\hat{D} \in \mathbb{R}^{m\times d}$. Here, we assume that $f_{\mathrm{u}}(x)$ is unknown and is parameterized as $f_{\mathrm{u}}(x) = \Theta f_{\mathrm{n}}(x)$, where $f_{\mathrm{n}}: \mathbb{R}^n \to \mathbb{R}^q$ and satisfies $f_{\mathrm{n}}^{\mathrm{T}}(x) f_{\mathrm{n}}(x) \le \gamma_{\mathrm{n}} x^{\mathrm{T}} x$, $x \in \mathbb{R}^n$, where $\gamma_{\mathrm{n}} > 0$, and $\Theta \in \mathbb{R}^{m\times q}$ is a matrix of uncertain constant parameters.

Next, to apply Theorem 1 to the uncertain system (1) with $f(x)$, $G(x)$ and D given by (24), let $K_{\mathrm{g}} \in \mathbb{R}^{m\times s}$, where $s = q + r$, be given by

$$K_{\mathrm{g}} = [\Theta_{\mathrm{n}} - \Theta,\ \Phi_{\mathrm{n}}] \tag{25}$$

where $\Theta_{\mathrm{n}} \in \mathbb{R}^{m\times q}$ and $\Phi_{\mathrm{n}} \in \mathbb{R}^{m\times r}$ are known matrices, and let

$$F(x) = \begin{bmatrix} f_{\mathrm{n}}(x) \\ \hat{f}_{\mathrm{n}}(x) \end{bmatrix} \tag{26}$$

where $\hat{f}_{\mathrm{n}}: \mathbb{R}^n \to \mathbb{R}^r$ satisfying $\hat{f}_{\mathrm{n}}^{\mathrm{T}}(x) \hat{f}_{\mathrm{n}}(x) \le \hat{\gamma}_{\mathrm{u}} x^{\mathrm{T}} x$, $x \in \mathbb{R}^n$, $\hat{\gamma}_{\mathrm{u}} > 0$, is an arbitrary function. In this case, it follows that, with $\hat{G}(x) = G_{\mathrm{s}}^{-1}(x)$

$$\begin{aligned} f_{\mathrm{s}}(x) &= f(x) + G(x)\hat{G}(x)K_{\mathrm{g}}F(x) \\ &= \tilde{A}x + \tilde{f}_{\mathrm{u}}(x) + \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(x) \end{bmatrix} G_{\mathrm{s}}^{-1}(x) \\ &\quad \times \left[\Theta_{\mathrm{n}} f_{\mathrm{n}}(x) - \Theta f_{\mathrm{n}}(x) + \Phi_{\mathrm{n}} \hat{f}_{\mathrm{n}}(x) \right] \\ &= \tilde{A}x + \begin{bmatrix} 0_{(n-m)\times 1} \\ \Theta_{\mathrm{n}} f_{\mathrm{n}}(x) + \Phi_{\mathrm{n}} \hat{f}_{\mathrm{n}}(x) \end{bmatrix} \end{aligned} \tag{27}$$

Note that, with $\hat{G}(x) = G_{\mathrm{s}}^{-1}(x)$, Ψ in Theorem 1 can be taken as $\Psi = \hat{D}$ so that $G(x)\hat{G}(x)\Psi = J(x) = D$ is satisfied, and (4) is satisfied with $\bar{\gamma} \ge \gamma_{\mathrm{n}} + \hat{\gamma}_{\mathrm{n}}$.

Now, since $\Theta_{\mathrm{n}} \in \mathbb{R}^{m\times q}$ and $\Phi_{\mathrm{n}} \in \mathbb{R}^{m\times r}$ are arbitrary constant matrices and $\hat{f}_{\mathrm{n}}: \mathbb{R}^n \to \mathbb{R}^r$ is an arbitrary function we can always construct K_{g}, $V_{\mathrm{s}}(x)$ and $F(x)$ without knowledge of $f(x)$ such that (2)–(5) hold. In particular, choosing $\Theta_{\mathrm{n}} f_{\mathrm{n}}(x) + \Phi_{\mathrm{n}} \hat{f}_{\mathrm{n}}(x) = \hat{A}x$, where $\hat{A} \in \mathbb{R}^{m\times n}$, it follows that (27) has the form $f_{\mathrm{s}}(x) = A_{\mathrm{s}}x$, where $A_{\mathrm{s}} = [A_0^{\mathrm{T}}, \hat{A}^{\mathrm{T}}]^{\mathrm{T}}$ is in multivariable controllable canonical form. Hence, in the case where $G(x) \equiv B$, choosing $\hat{A}$ such that A_{c} is asymptotically stable it follows that for sufficiently small $\varepsilon > 0$ there exists a positive-definite matrix P satisfying the following Riccati inequality

$$0 \ge A_{\mathrm{s}}^{\mathrm{T}} P A_{\mathrm{s}} - P + R + 4\varepsilon A_{\mathrm{s}}^{\mathrm{T}} P B B^{\mathrm{T}} P A_{\mathrm{s}} \tag{28}$$

where R is a positive-definite matrix. In this case, with $V_{\mathrm{s}}(x) = x^{\mathrm{T}} P x$, equations (2)–(5) are satisfied with $\hat{G}(x) \equiv I_m$, $P_{1u}(x) = 2x^{\mathrm{T}} A_{\mathrm{s}}^{\mathrm{T}} P B$, $P_{2u}(x) = B^{\mathrm{T}} P B$ and $\mu \le \lambda_{\min}(P)$, and hence the adaptive feedback controller (7) with update law (8) guarantees global asymptotic stability of the non-linear uncertain discrete-time dynamical system (1) where $f(x)$, $G(x)$ and $J(x)$ are given by (24) with $G_{\mathrm{s}}(x) \equiv B_{\mathrm{s}} \in \mathbb{R}^{m\times m}$. As mentioned above, it is important to note that it is not necessary to utilize a feedback linearizing function $F(x)$ to produce a linear $f_{\mathrm{s}}(x)$. However, when the system is in normal form, a feedback linearizing function $F(x)$ assures the existence of $V_{\mathrm{s}}(x)$ that satisfies the conditions (2)–(5).

It is important to note that by choosing $\Theta_{\mathrm{n}} = \Phi_{\mathrm{n}} = 0$ considerable simplification occurs in the update law. Specifically, in this case it follows that

$$G^{\dagger}(x) f_{\mathrm{s}}(x) = \left[0_{m\times(n-m)}, G_{\mathrm{s}}^{-1}(x) \right] \begin{bmatrix} A_0 \\ 0_{m\times n} \end{bmatrix} x = 0$$

and hence the update law (8) can be simplified as

$$\begin{aligned} K(k+1) = K(k) - \frac{1}{c + \tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)} Q \hat{G}^{-1}(x(k)) G^{\dagger}(x(k)) \\ \times x(k+1) \tilde{x}^{\mathrm{T}}(k) \end{aligned} \tag{29}$$

Finally, it is also important to note that Theorem 1 is not restricted to dynamical systems satisfying the linear growth constraint $f^{\mathrm{T}}(x) f(x) \le \hat{\gamma} x^{\mathrm{T}} x$, $x \in \mathbb{R}^n$, $\hat{\gamma} > 0$. Theorem 1 can be used to construct adaptive discrete-time controllers so long as the function $F(x)$ satisfies (4) and we can construct a function $f_{\mathrm{s}}(x)$ such that (3) holds.

Next, we consider the case where $f(x)$ and $G(x)$ are both uncertain. Specifically, we assume that $G(x)$ is such that $G_{\mathrm{s}}(x)$ is unknown and is parameterized as $G_{\mathrm{s}}(x) = B_{\mathrm{u}} G_{\mathrm{n}}(x)$, where $G_{\mathrm{n}}: \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is known and satisfies $\det G_{\mathrm{n}}(x) \ne 0$, $x \in \mathbb{R}^n$ and $B_{\mathrm{u}} \in \mathbb{R}^{m\times m}$, with $\det B_{\mathrm{u}} \ne 0$ and $\sigma_{\max}(B_{\mathrm{u}}) \le \alpha$, $\alpha > 0$, is an unknown symmetric sign-definite matrix but a bound α for the maximum singular value of B_{u} and the sign definiteness of B_{u} are known; that is, $B_{\mathrm{u}} > 0$ or $B_{\mathrm{u}} < 0$. For the statement of the next result define $B_0 \overset{\triangle}{=} [0_{m\times(n-m)}, I_m]^{\mathrm{T}}$ for $B_{\mathrm{u}} > 0$ and $B_0 \overset{\triangle}{=} [0_{m\times(n-m)}, -I_m]^{\mathrm{T}}$ for $B_{\mathrm{u}} < 0$.

Corollary 1: *Consider the non-linear system $\mathcal{G}$ given by* (1) *with $f(x)$, $G(x)$ and $J(x)$ given by* (24), *and $G_{\rm s}(x) = B_{\rm u}G_{\rm n}(x)$, where $B_{\rm u}$, with $\sigma_{\max}(B_{\rm u}) < \alpha$, $\alpha > 0$, is an unknown symmetric sign-definite matrix and the sign definiteness of $B_{\rm u}$ is known. Assume there exist a matrix $K_{\rm g} \in \mathbb{R}^{m\times s}$, functions $V_{\rm s}: \mathbb{R}^n \to \mathbb{R}$, $F: \mathbb{R}^n \to \mathbb{R}^s$, $P_{1u}: \mathbb{R}^n \to \mathbb{R}^{1\times m}$, $\ell: \mathbb{R}^n \to \mathbb{R}^t$, a non-negative-definite matrix function $P_{2u}: \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and positive constants $\bar{\gamma}$, ε, μ and ν such that $V_{\rm s}(\cdot)$ and $\ell(\cdot)$ are continuous, $V_{\rm s}(0) = 0$, $\ell(0) = 0$, $\hat{\alpha}^{-2}G_{\rm n}^{-{\rm T}}(x)P_{2u}(x)G_{\rm n}^{-1}(x) \le \nu I_m$, $x \in \mathbb{R}^n$, $\hat{\alpha} > \alpha/2$, and, for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, equations* (2)–(5) *hold. Then the adaptive feedback control law*

$$u(k) = \hat{\alpha}^{-1}G_{\rm n}^{-1}(x(k))K(k)\tilde{x}(k) \tag{30}$$

where $K(k) \in \mathbb{R}^{m\times(s+d)}$, $k \in \mathcal{N}$ and $\tilde{x}(k) \triangleq [F^{\rm T}(x(k)), w^{\rm T}(k)]^{\rm T}$, with update law

$$\begin{aligned} K(k+1) = K(k) - \frac{1}{c + \tilde{x}^{\rm T}(k)\tilde{x}(k)} B_0^{\rm T} \\ \times [x(k+1) - f_{\rm s}(x(k))]\tilde{x}^{\rm T}(k) \end{aligned} \tag{31}$$

guarantees that the solution $(x(k), K(k)) \equiv (0, [K_{\rm g}, -\Psi])$, where $\Psi \in \mathbb{R}^{m\times d}$, of the closed-loop system given by (1), (30) *and* (31) *is Lyapunov stable and $\ell(x(k)) \to 0$ as $k \to \infty$. If, in addition, $\ell^{\rm T}(x)\ell(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$.*

Proof: The result is a direct consequence of Theorem 1. First, let $\hat{G}(x) = \hat{\alpha}^{-1}G_{\rm n}^{-1}(x)$ and $\Psi = \hat{\alpha}B_{\rm u}^{-1}\hat{D}$ so that $G(x)\hat{G}(x) = [0_{m\times(n-m)}, \hat{\alpha}^{-1}B_{\rm u}]^{\rm T}$ and $G(x)\hat{G}(x)\Psi = D$, and let $K_{\rm g} = \hat{\alpha}B_{\rm u}^{-1}[\Theta_{\rm n} - \Theta, \Phi_{\rm n}]$. Next, since Q in (8) is an arbitrary positive-definite matrix with $\lambda_{\max}(Q) < 2$, it can be replaced by $\hat{\alpha}^{-1}|B_{\rm u}| = \hat{\alpha}^{-1}(B_{\rm u}^2)^{1/2}$, where $(\cdot)^{1/2}$ denotes the (unique) positive-definite square root. Now, since $B_{\rm u}$ is symmetric and sign definite it follows from the Schur decomposition that $B_{\rm u} = UD_{B_{\rm u}}U^{\rm T}$, where U is orthogonal and $D_{B_{\rm u}}$ is real diagonal. Hence, $\hat{\alpha}^{-1}|B_{\rm u}|\hat{G}^{-1}(x)G^{\dagger}(x) = [0_{m\times(n-m)}, \mathcal{I}_m] = B_0^{\rm T}$, where $\mathcal{I}_m = I_m$ for $B_{\rm u} > 0$ and $\mathcal{I}_m = -I_m$ for $B_{\rm s} < 0$. Now, equation (8) implies (31). □

3. Adaptive control for non-linear systems with ℓ_2 disturbances

In this section we consider the problem of characterizing adaptive feedback control laws for non-linear discrete-time uncertain dynamical systems with exogenous ℓ_2 disturbances. Specifically, we consider the controlled non-linear uncertain system $\mathcal{G}$ given by

$$\begin{aligned} x(k+1) = f(x(k)) + G(x(k))u(k) + J(x(k))w(k), \\ x(0) = x_0, \quad w(\cdot) \in \ell_2, \quad k \in \mathcal{N} \end{aligned} \tag{32}$$

with performance variables

$$z(k) = h(x(k)) \tag{33}$$

where $x(k) \in \mathbb{R}^n$, $k \in \mathcal{N}$, is the state vector, $u(k) \in \mathbb{R}^m$, $k \in \mathcal{N}$, is the control input, $w(k) \in \mathbb{R}^d$, $k \in \mathcal{N}$, is an unknown bounded energy ℓ_2 disturbance, $z(k) \in \mathbb{R}^p$, $k \in \mathcal{N}$, is a performance variable, $f: \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(0) = 0$, $G: \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $J: \mathbb{R}^n \to \mathbb{R}^{n\times d}$, and $h: \mathbb{R}^n \to \mathbb{R}^p$ is continuous and satisfies $h(0) = 0$. The following theorem generalizes Theorem 1 to discrete-time non-linear uncertain dynamical systems with exogenous ℓ_2 disturbances.

Theorem 2: *Consider the non-linear system $\mathcal{G}$ given by* (32) *and* (33). *Assume there exist a matrix $K_{\rm g} \in \mathbb{R}^{m\times s}$, functions $V_{\rm s}: \mathbb{R}^n \to \mathbb{R}$, $\hat{G}: \mathbb{R}^n \to \mathbb{R}^{m\times m}$, $F: \mathbb{R}^n \to \mathbb{R}^s$, $P_{1u}: \mathbb{R}^n \to \mathbb{R}^{1\times m}$, $P_{1w}: \mathbb{R}^n \to \mathbb{R}^{1\times d}$, $P_{uw}: \mathbb{R}^n \to \mathbb{R}^{m\times d}$, non-negative-definite matrix functions $P_{2u}: \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and $P_{2w}: \mathbb{R}^n \to \mathbb{R}^{d\times d}$, and positive constants $\bar{\gamma}$, $\hat{\delta}$, a, ε, μ and ν such that $V_{\rm s}(\cdot)$ is continuous and satisfies* (5), *$V_{\rm s}(0) = 0$,* $\det\hat{G}(x) \neq 0$, *$x \in \mathbb{R}^n$, $F(x)$ satisfies* (4), *$(\hat{G}^{-1}(x)G^{\dagger}(x)\times J(x))^{\rm T}(\hat{G}^{-1}(x)G^{\dagger}(x)J(x)) \le \hat{\delta}I_d$, $x \in \mathbb{R}^n$, $\hat{G}^{\rm T}(x)P_{2u}(x)\times\hat{G}(x) < \nu I_m$, $x \in \mathbb{R}^n$, and, for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^d$*

$$\begin{aligned} &V_{\rm s}(f(x) + G(x)u + J(x)w) \\ &\quad = V_{\rm s}(f(x)) + P_{1u}(x)u + u^{\rm T}P_{2u}(x)u + P_{1w}(x)w \\ &\qquad + u^{\rm T}P_{uw}(x)w + w^{\rm T}P_{2w}(x)w \end{aligned} \tag{34}$$

$$0 \ge V_{\rm s}(f_{\rm s}(x)) - V_{\rm s}(x) + \Gamma(x) + \varepsilon P_{1u}(x)\hat{G}(x)\hat{G}^{\rm T}(x)P_{1u}^{\rm T}(x) \tag{35}$$

$$\begin{aligned} \frac{a}{c + F^{\rm T}(x)F(x)}(2I - Q) \ge \frac{(1/4\varepsilon) + \nu}{1 + V_{\rm s}(x)} I_m \\ + \frac{1}{4\tilde{\lambda}}\tilde{P}_{uw}(x)\tilde{P}_{uw}^{\rm T}(x), \quad x \in \mathbb{R}^n \end{aligned} \tag{36}$$

where $f_{\rm s}(x)$ is given by (6)

$$\begin{aligned} \Gamma(x) \triangleq \tfrac{1}{4}P_{1w}(x)[(\gamma^2 - \tilde{\gamma}^2)I_m - P_{2w}(x)]^{-1}P_{1w}^{\rm T}(x) \\ + h^{\rm T}(x)h(x) \end{aligned} \tag{37}$$

$$\begin{aligned} \tilde{P}_{uw}(x) \triangleq &\frac{1}{1 + V_{\rm s}(x)}\hat{G}^{\rm T}(x)P_{uw}(x) \\ &- \frac{2a}{c + F^{\rm T}(x)F(x)}\hat{G}^{-1}(x)G^{\dagger}(x)J(x) \\ &+ \frac{2aF^{\rm T}(x)F(x)}{(c + F^{\rm T}(x)F(x))^2}Q\hat{G}^{-1}(x)G^{\dagger}(x)J(x) \end{aligned} \tag{38}$$

$\gamma > 0$, $(\gamma^2 - \tilde{\gamma}^2)I_m - P_{2w}(x) > 0$, $Q \in \mathbb{R}^{m\times m}$ is positive definite with $\lambda_{\max}(Q) < 2$ and $\tilde{\gamma}$ is such that

$$\frac{\tilde{\gamma}^2}{1 + V_{\rm s}(x)} - \frac{a\hat{\delta}\lambda_{\max}(Q)}{c + F^{\rm T}(x)F(x)} \ge \tilde{\lambda} > 0, \qquad x \in \mathbb{R}^n \tag{39}$$

where $\tilde{\lambda} \in \mathbb{R}$. *Then the adaptive feedback control law*

$$u(k) = \hat{G}(x(k))K(k)F(x(k)) \tag{40}$$

where $K(k) \in \mathbb{R}^{m \times s}$, $k \in \mathcal{N}$, *with update law*

$$\begin{aligned} K(k+1) = K(k) - &\frac{1}{c + F^{\mathrm{T}}(x(k))F(x(k))} \\ &\times Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k)) \\ &\times [x(k+1) - f_{\mathrm{s}}(x(k))]F^{\mathrm{T}}(k) \end{aligned} \tag{41}$$

guarantees that the solution $(x(k), K(k)) \equiv (0, K_{\mathrm{g}})$ *of the undisturbed* ($w(k) \equiv 0$) *closed-loop system given by* (32), (40) *and* (41) *is Lyapunov stable and* $h(x(k)) \to 0$ *as* $k \to \infty$. *If, in addition,* $h^{\mathrm{T}}(x)h(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, *then* $x(k) \to 0$ *as* $k \to \infty$ *for all* $x_0 \in \mathbb{R}^n$. *Furthermore, the solution* $x(k)$, $k \in \mathcal{N}$, *to the closed-loop system given by* (32), (40) *and* (41) *satisfies the non-expansivity constraint*

$$\sum_{k=0}^{\mathcal{K}-1} \frac{z^{\mathrm{T}}(k)z(k)}{1 + V_{\mathrm{s}}(x(k))} \leq \gamma^2 \sum_{k=0}^{\mathcal{K}-1} w^{\mathrm{T}}(k)w(k) + V(x(0), K(0))$$

$$\mathcal{K} \geq 0, \quad \gamma > 0, \quad w(\cdot) \in \ell_2 \tag{42}$$

where

$$V(x, K) \triangleq \ln(1 + V_{\mathrm{s}}(x)) + a\,\mathrm{tr}(K - K_{\mathrm{g}})^{\mathrm{T}} Q^{-1}(K - K_{\mathrm{g}}) \tag{43}$$

Proof: First, define $\tilde{K}(k) \triangleq K(k) - K_{\mathrm{g}}$, $\tilde{x}(k) = F(x(k))$ and $\tilde{u}(k) \triangleq \tilde{K}(k)\tilde{x}(k)$. Note that with $u(k)$, $k \in \mathcal{N}$, given by (40) it follows from (32) that

$$\begin{aligned} x(k+1) = &f(x(k)) + G(x(k))\hat{G}(x(k))K(k)F(x(k)) \\ &+ J(x(k))w(k), \quad x(0) = x_0, \quad w(\cdot) \in \ell_2, \quad k \in \mathcal{N} \end{aligned} \tag{44}$$

or, equivalently, using the definition for $f_{\mathrm{s}}(x)$ given in (6)

$$\begin{aligned} x(k+1) &= f_{\mathrm{s}}(x(k)) + G(x(k))\hat{G}(x(k))\tilde{K}(k)\tilde{x}(k) + J(x(k))w(k) \\ &= f_{\mathrm{s}}(x(k)) + G(x(k))\hat{G}(x(k))\tilde{u}(k) + J(x(k))w(k), \\ &\quad x(0) = x_0, \ w(\cdot) \in \ell_2, \ k \in \mathcal{N} \end{aligned} \tag{45}$$

Furthermore, note that by adding and subtracting K_{g} to and from (41) and using (45) it follows that

$$\begin{aligned} \tilde{K}(k+1) &= \tilde{K}(k) - \frac{1}{c + \tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)} Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k)) \\ &\quad \times [G(x(k))\hat{G}(x(k))\tilde{K}(k)\tilde{x}(k) + J(x(k))w(k)]\tilde{x}^{\mathrm{T}}(k) \\ &= \tilde{K}(k) - \frac{1}{c + \tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)} Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k) \\ &\quad - \frac{1}{c + \tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)} Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k)) \\ &\quad \times J(x(k))w(k)\tilde{x}^{\mathrm{T}}(k), \qquad k \in \mathcal{N} \end{aligned} \tag{46}$$

To show Lyapunov stability of the undisturbed closed-loop system (45) and (46) consider the Lyapunov function candidate given by (43). Note that $V(0, K_{\mathrm{g}}) = 0$ and, since $V_{\mathrm{s}}(\cdot)$ and Q are positive definite and $a > 0$, $V(x, K) > 0$ for all $(x, K) \neq (0, K_{\mathrm{g}})$. Furthermore, $V(x, K)$ is radially unbounded. Now, since (34) collapses to (2) in the case where $w(k) \equiv 0$, Lyapunov stability of the undisturbed closed-loop system (45) and (46) as well as $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$ follows as in the proof of Theorem 1.

To show that the non-expansivity constraint (42) holds, note that, for all $w \in \mathbb{R}^d$

$$\begin{aligned} 0 &\leq \left[\tfrac{1}{2}P_{1w}^{\mathrm{T}}(x) - ((\gamma^2 - \tilde{\gamma}^2)I_m - P_{2w}(x))w\right]^{\mathrm{T}} \\ &\quad \times [(\gamma^2 - \tilde{\gamma}^2)I_m - P_{2w}(x)]^{-1} \\ &\quad \times \left[\tfrac{1}{2}P_{1w}^{\mathrm{T}}(x) - ((\gamma^2 - \tilde{\gamma}^2)I_m - P_{2w}(x))w\right] \\ &= \Gamma(x) + (\gamma^2 - \tilde{\gamma}^2)w^{\mathrm{T}}w - z^{\mathrm{T}}z - P_{1w}(x)w \\ &\quad - w^{\mathrm{T}}P_{2w}(x)w \end{aligned} \tag{47}$$

Now, let $w(\cdot) \in \ell_2$ and let $x(k)$, $k \in \mathcal{N}$, denote the solution of the closed-loop system (45). Then, using (34), (35), (39), (46) and (47), the Lyapunov difference along the closed-loop system trajectories is given by

$$\begin{aligned} &\Delta V(x(k), K(k)) \\ &= \ln\Big(1 + V_{\mathrm{s}}(f_{\mathrm{s}}(x(k)) + G(x(k))u(k) + J(x(k))w(k))\Big) \\ &\quad + a\,\mathrm{tr}\Big(\tilde{K}(k) - \frac{1}{c + \tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)} Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k) \\ &\quad - \frac{1}{c + \tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)} Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))w(k)\tilde{x}^{\mathrm{T}}(k)\Big)^{\mathrm{T}} Q^{-1} \\ &\quad \times \Big(\tilde{K}(k) - \frac{1}{c + \tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)} Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k) \\ &\quad - \frac{1}{c + \tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)} Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))w(k)\tilde{x}^{\mathrm{T}}(k)\Big) \\ &\quad - \ln(1 + V_{\mathrm{s}}(x(k))) - a\,\mathrm{tr}\tilde{K}^{\mathrm{T}}(k)Q^{-1}\tilde{K}(k) \\ &= \ln\Big(1 + \Big[V_{\mathrm{s}}(f_{\mathrm{s}}(x(k))) + P_{1u}(x(k))\hat{G}(x(k))\tilde{u}(k) \\ &\quad + \tilde{u}^{\mathrm{T}}(k)\hat{G}^{\mathrm{T}}(x(k))P_{2u}(x(k))\hat{G}(x(k))\tilde{u}(k) + P_{1w}(x(k))w(k) \\ &\quad + \tilde{u}^{\mathrm{T}}(k)\hat{G}^{\mathrm{T}}(x(k))P_{uw}(x(k))w(k) + w^{\mathrm{T}}(k)P_{2w}(x(k))w(k) \\ &\quad - V_{\mathrm{s}}(x(k))\Big][1 + V_{\mathrm{s}}(x(k))]^{-1}\Big) + a\,\mathrm{tr}\tilde{K}^{\mathrm{T}}(k)Q^{-1}\tilde{K}(k) \end{aligned}$$

$$
\begin{aligned}
&+\frac{a}{(c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k))^2}\operatorname{tr}\tilde{x}(k)w^{\mathrm{T}}(k)J^{\mathrm{T}}(x(k)){G^{\dagger}}^{\mathrm{T}}(x(k))\hat{G}^{-\mathrm{T}}\\
&\times Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))w(k)\tilde{x}^{\mathrm{T}}(k)\\
&+\frac{a}{(c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k))^2}\operatorname{tr}\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k)\\
&-\frac{2a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\operatorname{tr}\tilde{K}^{\mathrm{T}}(k)\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k)\\
&-\frac{2a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\operatorname{tr}\tilde{K}^{\mathrm{T}}(k)\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))w(k)\tilde{x}^{\mathrm{T}}(k)\\
&+\frac{2a}{(c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k))^2}\operatorname{tr}\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))\\
&\times J(x(k))w(k)\tilde{x}^{\mathrm{T}}(k)-a\operatorname{tr}\tilde{K}^{\mathrm{T}}(k)Q^{-1}\tilde{K}(k)\\
\le{}&[-\Gamma(x(k))-\varepsilon P_{1u}(x(k))\hat{G}(x(k))\hat{G}^{\mathrm{T}}(x(k))P_{1u}^{\mathrm{T}}(x(k))\\
&+P_{1u}(x(k))\hat{G}(x(k))\tilde{u}(k)+\nu\tilde{u}^{\mathrm{T}}(k)\tilde{u}(k)+P_{1w}(x(k))w(k)\\
&+w^{\mathrm{T}}(k)P_{2w}(x(k))w(k)][1+V_{\mathrm{s}}(x(k))]^{-1}\\
&+\tilde{u}^{\mathrm{T}}(k)\Big[\frac{1}{1+V_{\mathrm{s}}(x(k))}\hat{G}^{\mathrm{T}}(x(k))P_{uw}(x(k))\\
&-\frac{2a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))\\
&+\frac{2a\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}{(c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k))^2}Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))\Big]w(k)\\
&+\frac{a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}w^{\mathrm{T}}(k)J^{\mathrm{T}}(x(k)){G^{\dagger}}^{\mathrm{T}}(x(k))\hat{G}^{-\mathrm{T}}Q\hat{G}^{-1}(x(k))\\
&\times G^{\dagger}(x(k))J(x(k))w(k)-\frac{a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\\
&\times\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)(2I_m-Q)\tilde{K}(k)\tilde{x}(k)\\
\le{}&\Big[\gamma^2w^{\mathrm{T}}(k)w(k)-z^{\mathrm{T}}(k)z(k)-\varepsilon P_{1u}(x(k))\hat{G}(x(k))\hat{G}^{\mathrm{T}}(x(k))\\
&\times P_{1u}^{\mathrm{T}}(x(k))+P_{1u}(x(k))\hat{G}(x(k))\tilde{u}(k)+\nu\tilde{u}^{\mathrm{T}}(k)\tilde{u}(k)\Big]\\
&\times[1+V_{\mathrm{s}}(x(k))]^{-1}+\tilde{u}^{\mathrm{T}}(k)\tilde{P}_{uw}(x(k))w(k)-\tilde{\lambda}w^{\mathrm{T}}(k)w(k)\\
&-\frac{a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)(2I_m-Q)\tilde{K}(k)\tilde{x}(k),\quad k\in\mathcal{N}
\end{aligned}
\tag{48}
$$

where in (48) we used $\ln a-\ln b=\ln(a/b)$ and $\ln(1+c)\le c$ for $a,b>0$ and $c\ge -1$, respectively, and $\tilde{x}^{\mathrm{T}}\tilde{x}/(c+\tilde{x}^{\mathrm{T}}\tilde{x})<1$. Now, using (36), adding and subtracting

$$
\tilde{u}^{\mathrm{T}}(k)\Big[\frac{1}{4\varepsilon}\frac{1}{1+V_{\mathrm{s}}(x(k))}I_m+\frac{1}{4\tilde{\lambda}}\tilde{P}_{uw}(x(k))\tilde{P}_{uw}^{\mathrm{T}}(x(k))\Big]\tilde{u}(k),\qquad k\in\mathcal{N}
$$

to and from (48), and collecting terms yields

$$
\begin{aligned}
&\Delta V(x(k),K(k))\\
&\le\frac{1}{1+V_{\mathrm{s}}(x(k))}[\gamma^2w^{\mathrm{T}}(k)w(k)-z^{\mathrm{T}}(k)z(k)]\\
&-\frac{1}{1+V_{\mathrm{s}}(x(k))}\big[P_{1u}(x(k)),\tilde{u}^{\mathrm{T}}(k)\big]\\
&\times\begin{bmatrix}\varepsilon\hat{G}(x(k))\hat{G}^{\mathrm{T}}(x(k)) & -\frac{1}{2}\hat{G}(x(k))\\ -\frac{1}{2}\hat{G}^{\mathrm{T}}(x(k)) & (1/4\varepsilon)I_m\end{bmatrix}\begin{bmatrix}P_{1u}^{\mathrm{T}}(x(k))\\ \tilde{u}(k)\end{bmatrix}\\
&-\big[\tilde{u}^{\mathrm{T}}(k),w^{\mathrm{T}}(k)\big]\\
&\times\begin{bmatrix}(1/4\tilde{\lambda})\tilde{P}_{uw}(x(k))\tilde{P}_{uw}^{\mathrm{T}}(x(k)) & -\frac{1}{2}\tilde{P}_{uw}(x(k))\\ -\frac{1}{2}\tilde{P}_{uw}^{\mathrm{T}}(x(k)) & \tilde{\lambda}I_d\end{bmatrix}\begin{bmatrix}\tilde{u}(k)\\ w(k)\end{bmatrix}\\
&+\tilde{u}^{\mathrm{T}}(k)\Big[\frac{1}{4\varepsilon}\frac{1}{1+V_{\mathrm{s}}(x(k))}I_m+\frac{1}{4\tilde{\lambda}}\tilde{P}_{uw}(x(k))\tilde{P}_{uw}^{\mathrm{T}}(x(k))\Big]\tilde{u}(k)\\
&+\frac{\nu}{1+V_{\mathrm{s}}(x(k))}\tilde{u}^{\mathrm{T}}(k)\tilde{u}(k)-\frac{a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\\
&\times\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)(2I-Q)\tilde{K}(k)\tilde{x}(k)\\
\le{}&\gamma^2w^{\mathrm{T}}(k)w(k)-\frac{z^{\mathrm{T}}(k)z(k)}{1+V_{\mathrm{s}}(x(k))}\\
&-\tilde{u}^{\mathrm{T}}(k)\Big[\frac{a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}(2I-Q)-\frac{(1/4\varepsilon)+\nu}{1+V_{\mathrm{s}}(x(k))}I_m\\
&\qquad-\frac{1}{4\tilde{\lambda}}\tilde{P}_{uw}(x(k))\tilde{P}_{uw}^{\mathrm{T}}(x(k))\Big]\tilde{u}(k)\\
\le{}&\gamma^2w^{\mathrm{T}}(k)w(k)-\frac{z^{\mathrm{T}}(k)z(k)}{1+V_{\mathrm{s}}(x(k))},\quad k\in\mathcal{N}
\end{aligned}
\tag{49}
$$

Now, summing (49) over $k=0,\ldots,\mathcal{K}-1$ yields

$$
V(x(\mathcal{K}),K(\mathcal{K}))\le\sum_{k=0}^{\mathcal{K}-1}\left[\gamma^2w^{\mathrm{T}}(k)w(k)-\frac{z^{\mathrm{T}}(k)z(k)}{1+V_{\mathrm{s}}(x(k))}\right]+V(x(0),K(0)),\quad\mathcal{K}\ge 0,\quad w(\cdot)\in\ell_2
\tag{50}
$$

which, by noting that $V(x(\mathcal{K}),K(\mathcal{K}))\ge 0$, $\mathcal{K}\ge 0$, yields (42). □

It is important to note that unlike Theorem 1 requiring a matching condition on the disturbance, Theorem 2 does not require any such matching condition. Furthermore, as shown in § 2, if (32) is in normal form with asymptotically stable internal dynamics and $f^{\mathrm{T}}(x)f(x)\le\hat{\gamma}x^{\mathrm{T}}x$, $x\in\mathbb{R}^n$, where $\hat{\gamma}>0$, then we can always construct a function $F\colon\mathbb{R}^n\to\mathbb{R}^s$ such that $F(\cdot)$ satisfies (4) and (34)–(36) hold without requiring knowledge of the system dynamics. In addition, in the case where $J(x)=D$ and $h(x)=Ex$, the adaptive controller (40) can be verified to guarantee the modified non-expansivity constraint (42) using standard *linear* H_∞ methods. Specifically, choosing $f_{\mathrm{s}}(x)=A_{\mathrm{s}}x$, where A_{s} is asymptotically stable and in multivariable controllable canonical form, it follows from standard discrete-time H_∞ theory (Gu *et al.* 1989) that if (A_{s},E) is observable,

$\|G(s)\|_\infty < \sqrt{\gamma^2 - \tilde{\gamma}^2}$, where $G(s) = E(sI_n - A_{\rm s})^{-1}D$, if and only if there exists a positive-definite matrix P satisfying the discrete-time bounded real Riccati inequality

$$0 > A_{\rm s}^{\rm T}PA_{\rm s} - P + A_{\rm s}^{\rm T}PD[(\gamma^2 - \tilde{\gamma}^2)I_m - D^{\rm T}PD]^{-1} \times D^{\rm T}PA_{\rm s} + E^{\rm T}E \tag{51}$$

In this case, if $G(x) \equiv B$ is a constant matrix, then there exists a sufficiently small $\varepsilon > 0$ such that

$$0 \geq A_{\rm s}^{\rm T}PA_{\rm s} - P + A_{\rm s}^{\rm T}PD[(\gamma^2 - \tilde{\gamma}^2)I_m - D^{\rm T}PD]^{-1}D^{\rm T}PA_{\rm s} + E^{\rm T}E + 4\varepsilon A_{\rm s}^{\rm T}PBB^{\rm T}PA_{\rm s} \tag{52}$$

Now, with $V_{\rm s}(x) = x^{\rm T}Px$, there exists $\tilde{\lambda} > 0$ such that (34)–(36) and (39) are satisfied with $\hat{G}(x) = I_m$, $Q = I_m$, $P_{1u}(x) = 2x^{\rm T}A_{\rm s}^{\rm T}PB, P_{2u}(x) = B^{\rm T}PB, P_{1w}(x) = 2x^{\rm T}A_{\rm s}^{\rm T}PD$, $P_{uw}(x) = 2B^{\rm T}PD$, $P_{2w}(x) = D^{\rm T}PD$ and

$$a > \left(\frac{1}{4\varepsilon} + \nu\right)\max\left\{c,\ \frac{\bar{\gamma}}{\mu}\right\}$$

Hence, the adaptive feedback controller (40) with update law (41), or, equivalently

$$K(k+1) = K(k) - \frac{1}{c + F^{\rm T}(x(k))F(x(k))}B^{\dagger} \times [x(k+1) - A_{\rm s}\,x(k)]F^{\rm T}(k) \tag{53}$$

guarantees global asymptotic stability of the non-linear undisturbed ($w(k) \equiv 0$) dynamical system (32), where $f(x)$ and $G(x)$ are given by (24) with $G_{\rm s}(x) \equiv B_{\rm s}$. Furthermore, the solution $x(k)$, $k \in \mathcal{N}$, of the closed-loop *non-linear* dynamical system (32), (33) is guaranteed to satisfy the non-expansivity constraint (42).

Finally, if $f(x)$ and $G(x)$ given by (24) are uncertain and $G_{\rm s}(x) = B_{\rm u}G_{\rm n}(x)$, where a bound for the maximum singular value α of $B_{\rm u}$ and the sign definiteness of $B_{\rm u}$ are known, then using an identical approach as in § 2, it can be shown that the adaptive feedback control law

$$u(k) = \hat{\alpha}^{-1}G_{\rm n}^{-1}(x(k))K(k)F(x(k)) \tag{54}$$

where $\hat{\alpha} > \alpha/2$, with update law

$$K(k+1) = K(k) - \frac{1}{c + \tilde{x}^{\rm T}(k)\tilde{x}(k)}B_0^{\rm T} \times [x(k+1) - f_{\rm s}(x(k))]\tilde{x}^{\rm T}(k) \tag{55}$$

where B_0 is defined as in § 2, guarantees asymptotic stability and non-expansivity of (32).

4. Illustrative numerical examples

In this section we present three numerical examples to demonstrate the utility of the proposed discrete-time adaptive control framework for adaptive stabilization, disturbance rejection and command following.

Example 1: Consider the linear uncertain system given by

$$z(k+2) + a_1z(k+1) + a_0z(k) = bu(k) + \hat{d}\sin 7k,$$
$$z(0) = z_0, \quad z(1) = z_1, \quad k \in \mathcal{N} \tag{56}$$

where $z(k) \in \mathbb{R}$, $k \in \mathcal{N}$, $u(k) \in \mathbb{R}$, $k \in \mathcal{N}$, and a_0, a_1, $b, \hat{d} \in \mathbb{R}$ are unknown constants. Note that with $x_1(k) = z(k)$ and $x_2(k) = z(k+1)$, equation (56) can be written in state space form (1) with $x = [x_1, x_2]^{\rm T}$, $f(x) = [x_2, -a_0x_1 - a_1x_2]^{\rm T}$, $G(x) = [0, b]^{\rm T}$, $J(x) = [0, \hat{d}]^{\rm T}$ and $w(k) = \sin 7k$. Here, we assume that $f(x)$ is unknown and can be parameterized as $f(x) = [x_2,\ \theta_1x_1 + \theta_2x_2]^{\rm T}$, where θ_1 and θ_2 are unknown constants. Furthermore, we assume that sgn b is known and $|b| < \alpha = 2$. Next, let9 $G_{\rm n}(x) = 1$, $F(x) = x$, $\hat{\alpha} = 1$, and $K_{\rm g} = (1/b)[\theta_{{\rm n}_1} - \theta_1,\ \theta_{{\rm n}_2} - \theta_2]$, where $\theta_{{\rm n}_1}$, $\theta_{{\rm n}_2}$ are arbitrary scalars, so that

$$f_{\rm s}(x) = f(x) + \begin{bmatrix} 0 \\ b \end{bmatrix}\frac{1}{b}\left[\theta_{{\rm n}_1} - \theta_1, \theta_{{\rm n}_2} - \theta_2\right]F(x) = \begin{bmatrix} 0 & 1 \\ \theta_{{\rm n}_1} & \theta_{{\rm n}_2} \end{bmatrix}x$$

Note that since (56) is linear all the conditions of Corollary 1 are trivially satisfied. Now, with the proper choice of $\theta_{{\rm n}_1}$ and $\theta_{{\rm n}_2}$, it follows from Corollary 1 that the adaptive feedback controller (30) guarantees that $x(k) \to 0$ as $k \to \infty$. With $\hat{\alpha} = 1$, $\theta_1 = -1$, $\theta_2 = 0.25$, $b = 0.4$, $\hat{d} = 10$, $c = 1$, $\theta_{{\rm n}_1} = -0.02$, $\theta_{{\rm n}_2} = 0.3$ and initial conditions $x(0) = [-1, 3]^{\rm T}$ and $K(0) = [0, 0, 0]$, figure 1 shows the phase portrait of the controlled and uncontrolled system. Note that the adaptive controller is switched on at $k = 30$. Figure 2 shows the state trajectories vs. time and the control signal vs. time. Finally, figure 3 shows the adaptive gain history vs. time.

Example 2: Consider the two-degree of freedom uncertain linear system given by

$$M_{\rm s}z(k+2) + C_{\rm s}z(k+1) + K_{\rm s}z(k) = u(k),$$
$$z(0) = z_0, \quad z(1) = z_1, \quad k \in \mathcal{N} \tag{57}$$

where $z(k) \in \mathbb{R}^2$, $u(k) \in \mathbb{R}^2$, $k \in \mathcal{N}$, and $M_{\rm s}, C_{\rm s}, K_{\rm s} \in \mathbb{R}^{2\times 2}$ are unknown matrices. Here we assume that $M_{\rm s} = M_{\rm s}^{\rm T} > 0$ and $\lambda_{\max}(M_{\rm s}^{-1}) < \alpha = 2$ but otherwise $M_{\rm s}$ is unknown. Let $r_{\rm d}(k)$ be a desired command signal and define the error state $\tilde{e}(k) \stackrel{\Delta}{=} z(k) - r_{\rm d}(k)$ so that the error dynamics are given by

$$M_{\rm s}\tilde{e}(k+2) + C_{\rm s}\tilde{e}(k+1) + K_{\rm s}\tilde{e}(k) = u(k) - M_{\rm s}r_{\rm d}(k+2) - C_{\rm s}r_{\rm d}(k+1) - K_{\rm s}r_{\rm d}(k),$$
$$\tilde{e}(0) = \tilde{e}_0, \quad \tilde{e}(1) = \tilde{e}_1, \quad k \in \mathcal{N} \tag{58}$$

Note that with $e_1(k) = \tilde{e}(k)$ and $e_2(k) = \tilde{e}(k+1)$, equation (58) can be written in state space form (22) with

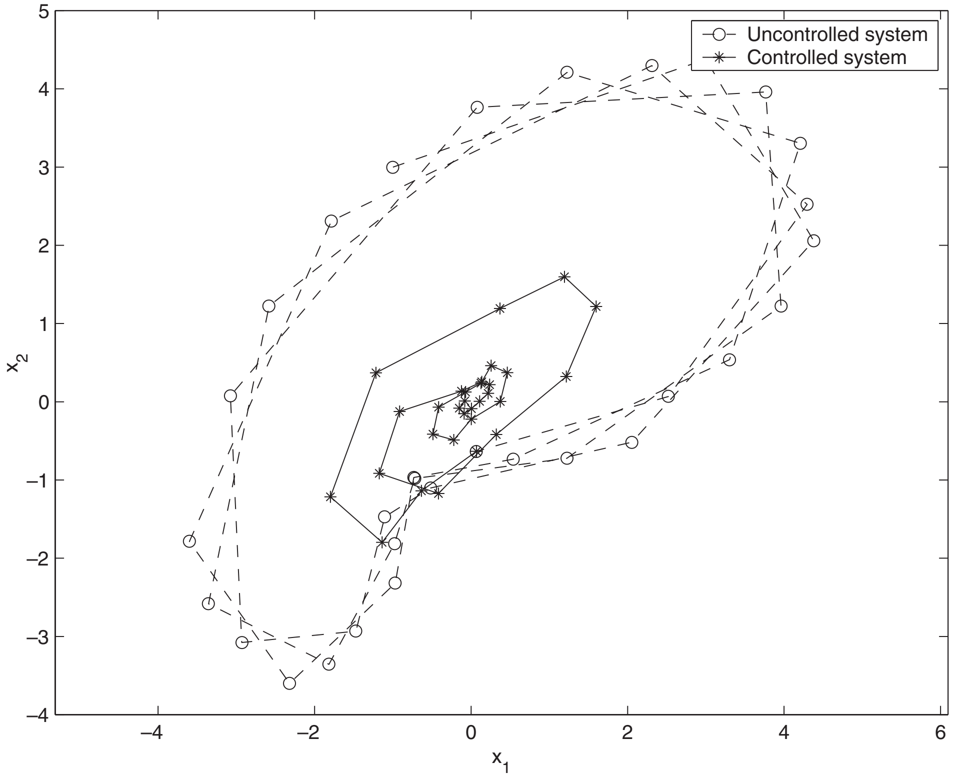

Figure 1. Phase portrait of controlled and uncontrolled system.

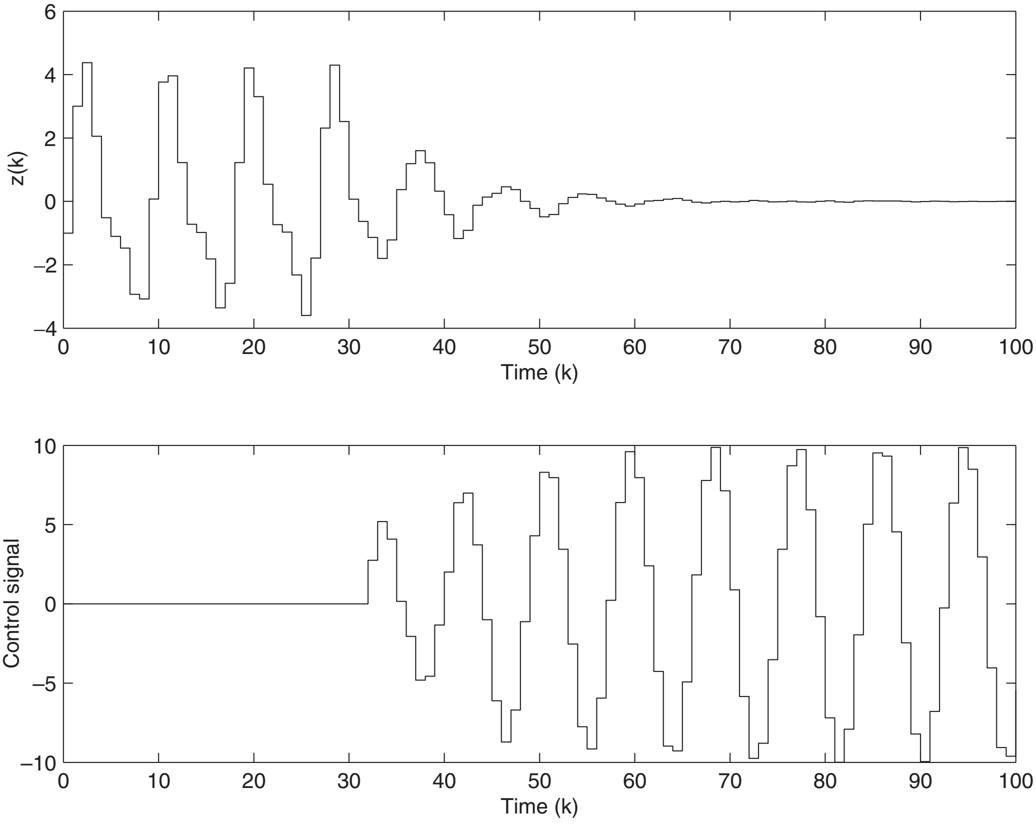

Figure 2. State trajectories and control signal vs. time.

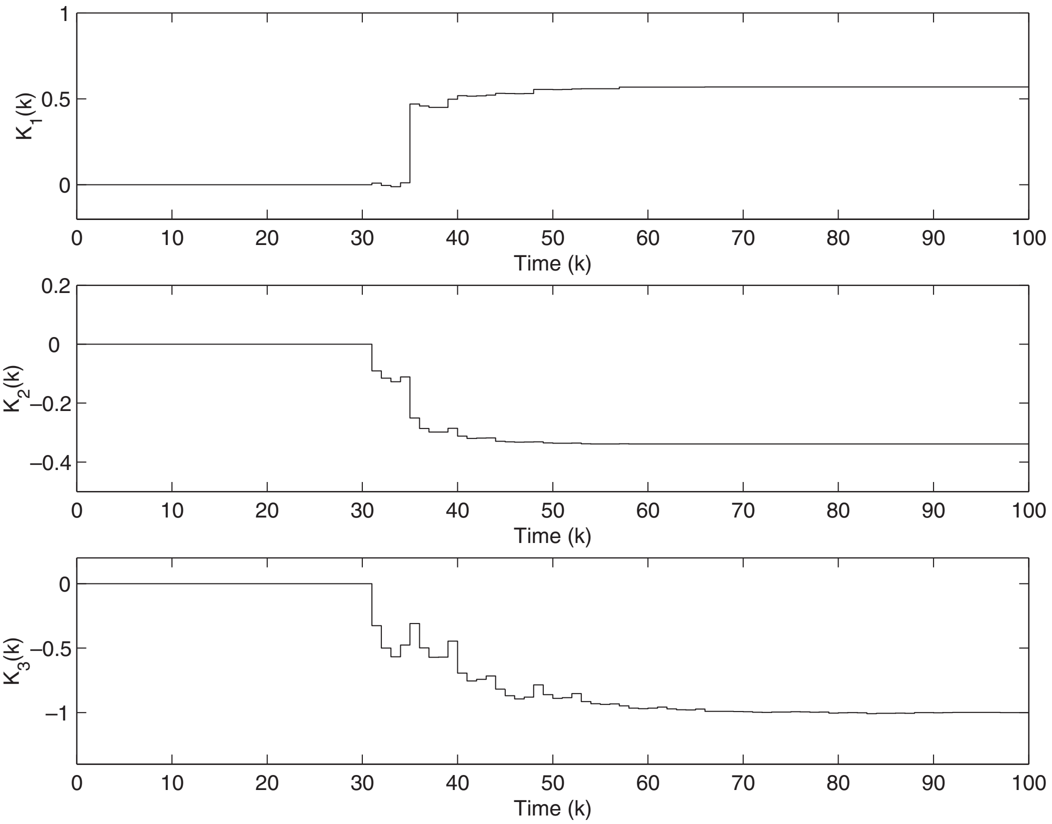

Figure 3. Adaptive gain history vs. time.

$e=[e_1^{\mathrm T}, e_2^{\mathrm T}]^{\mathrm T}$, $f_{\mathrm k}(k,e)=[e_2^{\mathrm T}, -(M_{\mathrm s}^{-1}K_{\mathrm s}e_1+M_{\mathrm s}^{-1}C_{\mathrm s}e_2)^{\mathrm T}]^{\mathrm T}$, $G(k,e)=[0_{2\times 2}, M_{\mathrm s}^{-1}]^{\mathrm T}$, $J_{\mathrm k}(k,e)=[0_{6\times 2}, \hat{D}_{\mathrm k}^{\mathrm T}]^{\mathrm T}$, where $\hat{D}_{\mathrm k}=[-I_2, -M_{\mathrm s}^{-1}C_{\mathrm s}, -M_{\mathrm s}^{-1}K_{\mathrm s}]$ and $w_{\mathrm k}(k)=[r_{\mathrm d}^{\mathrm T}(k+2), r_{\mathrm d}^{\mathrm T}(k+1), r_{\mathrm d}^{\mathrm T}(k)]^{\mathrm T}$. Next, let $G_{\mathrm n}(x)=I$, $F(e)=e$, $\hat{\alpha}=1$ and $K_{\mathrm g}=M_{\mathrm s}[\Theta_{\mathrm n_1}+M_{\mathrm s}^{-1}K_{\mathrm s}, \Theta_{\mathrm n_2}+M_{\mathrm s}^{-1}C_{\mathrm s}]$, where $\Theta_{\mathrm n_1}\in\mathbb{R}^{2\times 2}$, $\Theta_{\mathrm n_2}\in\mathbb{R}^{2\times 2}$ are arbitrary matrices, so that

$$f_{\mathrm s}(e)=\begin{bmatrix} 0_2 & I_2 \\ \Theta_{\mathrm n_1} & \Theta_{\mathrm n_2}\end{bmatrix}e$$

Note that since (57) is linear all the conditions of Corollary 1 are trivially satisfied. Now, with the proper choice of $\Theta_{\mathrm n_1}$ and $\Theta_{\mathrm n_2}$, it follows from Corollary 1 and Remark 3 that the adaptive feedback controller (30) guarantees that $e(k)\to 0$ as $t\to\infty$. With

$$M_{\mathrm s}=\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix},\qquad C_{\mathrm s}=\begin{bmatrix}2 & 2\\ 1 & 1\end{bmatrix},\qquad K_{\mathrm s}=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$r_{\mathrm d}(k)=[\sin 0.5k,\ 0.5]^{\mathrm T}, \hat{\alpha}=1, c=1, \Theta_{\mathrm n_1}=\Theta_{\mathrm n_2}=0_2$, and initial conditions $x(0)=[3, -4, -2, 1]^{\mathrm T}$ and $K(0)=0_{2\times 10}$, figure 4 shows the actual positions and the reference signals vs. time and the control signals vs. time. Note that the adaptive controller is switched on at $k=40$.

Example 3: Consider the non-linear uncertain system given by

$$z(k+2)+a_1\frac{z^3(k)}{1+z^2(k)}+a_2\ln(1+|z(k+1)|)=bu(k),$$
$$z(0)=z_0,\quad z(1)=z_1,\quad k\in\mathcal{N}\qquad(59)$$

where $z(k)\in\mathbb{R}$, $k\in\mathcal{N}$, $u(k)\in\mathbb{R}$, $k\in\mathcal{N}$, and a_1, a_2, $b\in\mathbb{R}$ are unknown constants. Note that with $x_1(k)=z(k)$ and $x_2(k)=z(k+1)$, equation (59) can be written in state space form (1) with $x=[x_1, x_2]^{\mathrm T}$

$$f(x)=\left[x_2,\ -a_1\frac{x_1^3}{1+x_1^2}-a_2\ln(1+|x_2|)\right]^{\mathrm T}$$

$G(x)=[0, b]^{\mathrm T}$ and $w(k)\equiv 0$. Here, we assume that $f(x)$ is unknown and can be parameterized as

$$f(x)=\left[x_2,\ \theta_1\frac{x_1^3}{1+x_1^2}+\theta_2\ln(1+|x_2|)\right]^{\mathrm T}$$

where θ_1 and θ_2 are unknown constants. Furthermore, we assume that $\operatorname{sgn} b$ is known and $|b|<\alpha=2$. Next, let $G_{\mathrm n}(x)=1$

$$F(x)=\left[\frac{x_1^3}{1+x_1^2},\ \ln(1+|x_2|),\ x^{\mathrm T}\right]^{\mathrm T}$$

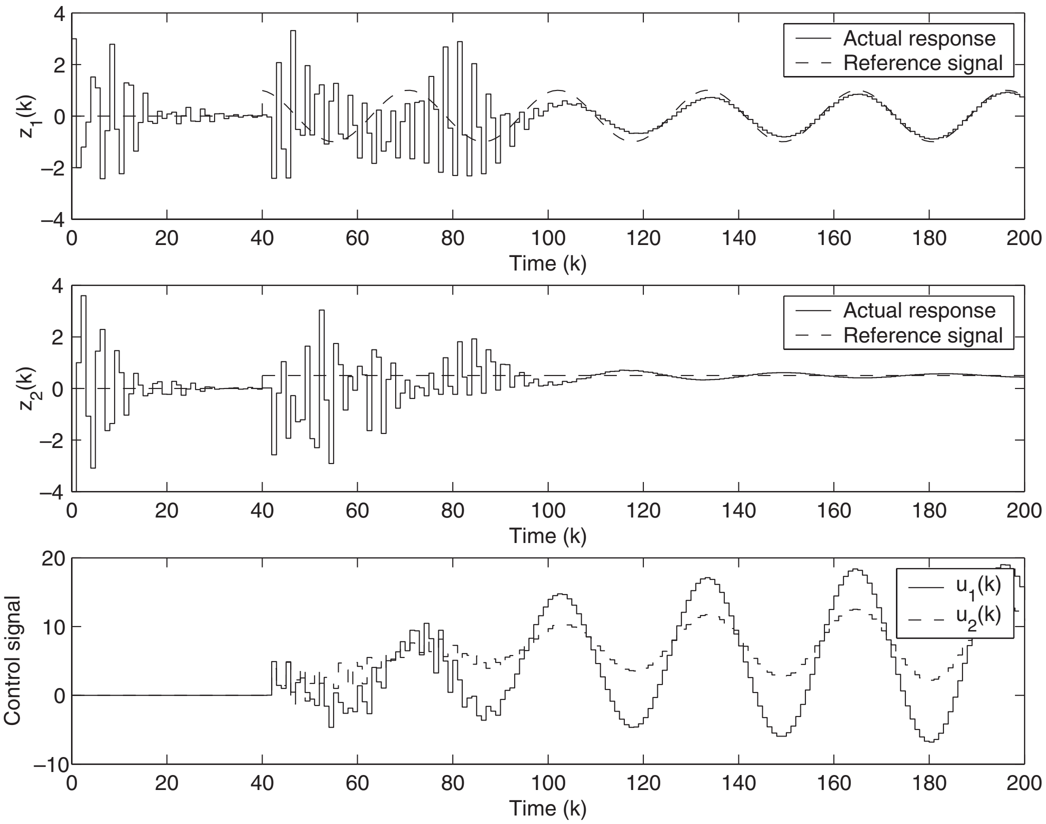

Figure 4. Positions and control signals vs. time.

$\hat{\alpha} = 1$ and $K_g = (1/b)[-\theta_1, -\theta_2, \phi_{n_1}, \phi_{n_2}]$, where ϕ_{n_1}, ϕ_{n_2} are arbitrary scalars, so that

$$f_s(x) = f(x) + \begin{bmatrix} 0 \\ b \end{bmatrix} \frac{1}{b} \left[-\theta_1, -\theta_2, \phi_{n_1}, \phi_{n_2} \right] F(x)$$

$$= \begin{bmatrix} 0 & 1 \\ \phi_{n_1} & \phi_{n_2} \end{bmatrix} x$$

In addition, note that

$$F^{\mathrm{T}}(x)F(x) = \left(\frac{x_1^2}{1 + x_1^2} \right)^2 x_1^2 + \ln^2(1 + |x_2|) + x^{\mathrm{T}} x \le 2x^{\mathrm{T}} x$$

and thus (4) is satisfied with $\bar{\gamma} = 2$. Now, with the proper choice of ϕ_{n_1} and ϕ_{n_2}, it follows from Corollary 1 that the adaptive feedback controller (30) guarantees that $x(k) \to 0$ as $k \to \infty$. With $\hat{\alpha} = 1$, $\theta_1 = 2$, $\theta_2 = -3$, $b = 1.4$, $c = 1$, $\theta_{n_1} = 0.1$, $\theta_{n_2} = 0.1$, and initial conditions $x(0) = [1.5, 7.3]^{\mathrm{T}}$ and $K(0) = [0, 0, 0, 0]$, figure 5 shows the state trajectory vs. time and the control signal vs. time. Finally, figure 6 shows the adaptive gain history vs. time.

5. Conclusion

A discrete-time direct adaptive non-linear control framework for adaptive stabilization, disturbance rejection and command following of multivariable non-linear uncertain dynamical systems with exogenous bounded disturbances and bounded energy ℓ_2 disturbances was developed. This framework is distinct from the standard discrete-time adaptive control methods for model reference and tracking problems developed in the literature predicated on the classical key technical lemma and quadratic Lyapunov functions, which does not guarantee Lyapunov stability. Specifically, using logarithmic Lyapunov functions the proposed framework was shown to guarantee partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Hence, unlike continuous-time adaptive control theory based on quadratic Lyapunov functions, logarithmic Lyapunov functions are shown to be essential for discrete-time Lyapunov-based adaptive control. Furthermore, in the case where the non-linear system is represented in normal form, the non-linear adaptive controllers were constructed without knowledge of the system dynamics. Future research will involve using logarithmic Lyapunov functions to extend discrete-time adaptive control results based on recursive least squares and least mean squares algorithms to additionally guarantee partial asymptotic stability. Finally, output feedback extensions will also be considered.

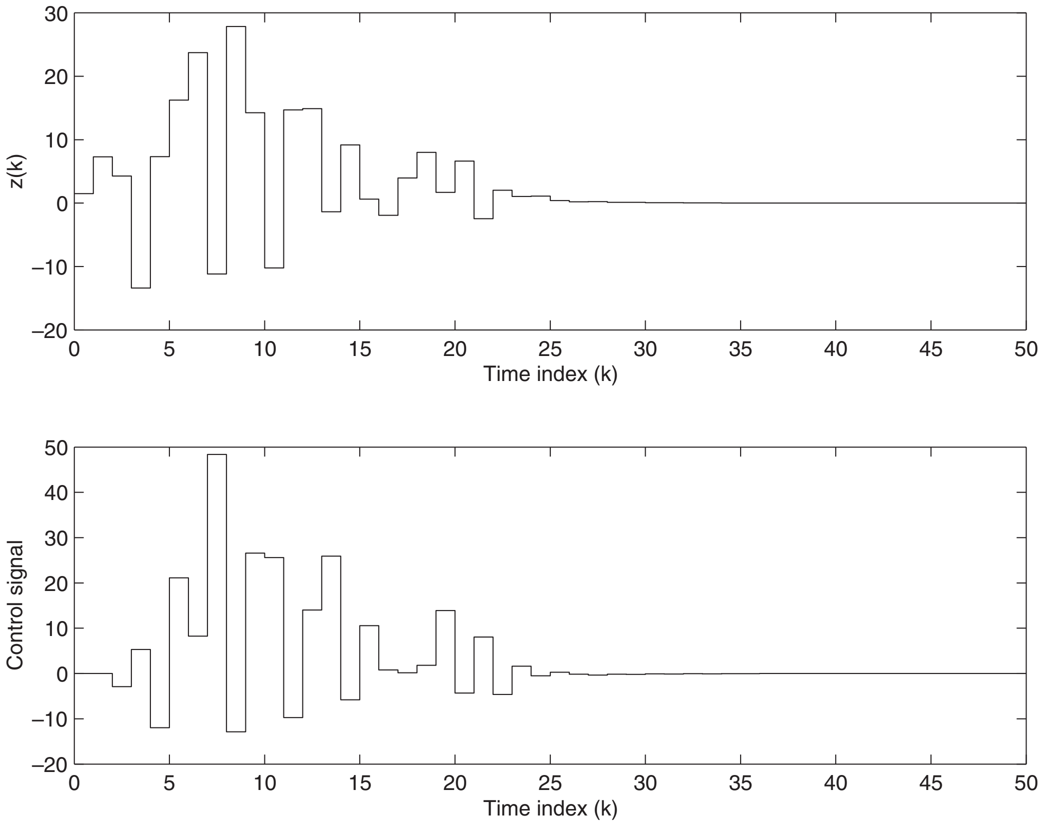

Figure 5. State trajectory and control signal vs. time.

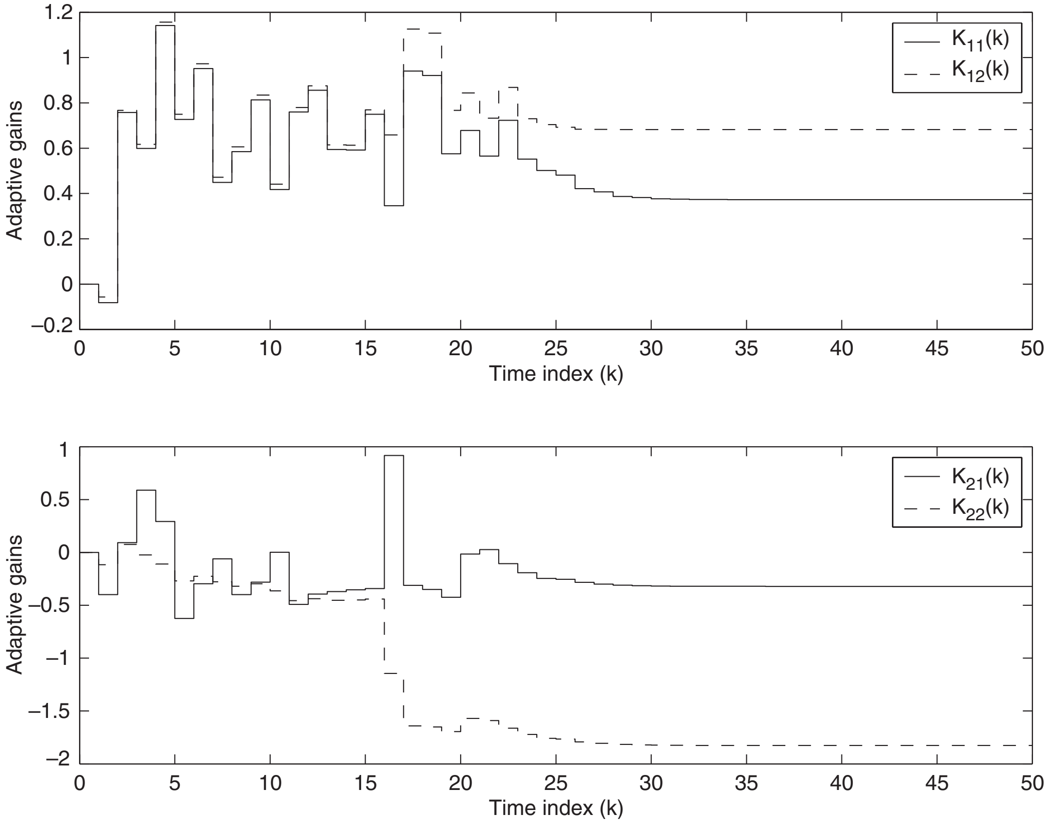

Figure 6. Adaptive gain history vs. time.

Acknowledgement

This research was supported in part by the Air Force Office of Scientific Research under Grant F49620-00-1-0095.

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