

Direct adaptive control for non-linear uncertain systems with exogenous disturbances

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SUMMARY

A direct adaptive non-linear control framework for multivariable non-linear uncertain systems with exogenous bounded disturbances is developed. The adaptive non-linear controller addresses adaptive stabilization, disturbance rejection and adaptive tracking. The proposed framework is Lyapunov-based and guarantees partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. In the case of bounded energy L_2 disturbances the proposed approach guarantees a non-expansivity constraint on the closed-loop input–output map. Finally, several illustrative numerical examples are provided to demonstrate the efficacy of the proposed approach. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: adaptive non-linear control; non-linear uncertain systems; adaptive stabilization; adaptive tracking; disturbance rejection; L_2 disturbances

1. INTRODUCTION

Unavoidable discrepancies between system models and real-world systems can result in degradation of control-system performance including instability. Thus, it is not surprising that one of the fundamental problems in feedback control design is the ability of the control system to guarantee robustness with respect to system uncertainties in the design model. To this end, adaptive control along with robust control theory have been developed to address the problem of system uncertainty in control-system design. The fundamental differences between adaptive control design and robust control theory can be traced to the modelling and treatment of system uncertainties as well as the controller architecture structures. In particular, adaptive control [1–4] is based on constant linearly parameterized system uncertainty models of a known structure but unknown variation, while robust control [5,6] is predicated on structured and/or unstructured linear or non-linear (possibly time-varying) operator uncertainty models consisting

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of bounded variation. Hence, for systems with constant real parameter uncertainty, robust controllers will unnecessarily sacrifice performance whereas adaptive feedback controllers can tolerate far greater system uncertainty levels to improve system performance. Furthermore, in contrast to fixed-gain robust controllers, which maintain specified constants within the feedback control law to *sustain* robust performance, adaptive controllers directly or indirectly adjust feedback gains to maintain closed-loop stability and *improve* performance in the face of system uncertainties. Specifically, indirect adaptive controllers utilize parameter update laws to identify unknown system parameters and adjust feedback gains to account for system variation, while direct adaptive controllers directly adjust the controller gains in response to plant variations.

In this paper we develop a direct adaptive control framework for adaptive stabilization, disturbance rejection, and command following of multivariable non-linear uncertain systems with exogenous disturbances. In particular, in the first part of the paper, a Lyapunov-based direct adaptive control framework is developed that requires a matching condition on the system disturbance and guarantees partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Furthermore, the remainder of the state associated with the adaptive controller gains is shown to be Lyapunov stable. In the case where the non-linear system is represented in normal form [7] with input-to-state stable zero dynamics [7,8], we construct non-linear adaptive controllers *without* requiring knowledge of the system dynamics or the system disturbance. In addition, the proposed non-linear adaptive controllers also guarantee asymptotic stability of the system state if the system dynamics are unknown *and* the input matrix function is parameterized by an unknown constant sign definite matrix. Finally, in the second part of the paper, we generalize the aforementioned results to uncertain non-linear systems with exogenous L_2 disturbances. In this case, we remove the matching condition on the system disturbance. In addition, the proposed framework guarantees that the closed-loop non-linear input-output map from uncertain exogenous L_2 disturbances to system performance variables is non-expansive (gain bounded) and the solution of the closed-loop system is partially asymptotically stable. The proposed adaptive controller thus addresses the problem of disturbance rejection for non-linear uncertain systems with bounded energy (square-integrable) L_2 signal norms on the disturbances and performance variables. This is clearly relevant for uncertain systems with poorly modelled disturbances which possess significant power within arbitrarily small bandwidths.

We emphasize that the direct adaptive stabilization framework developed in this paper is distinct from the methods given in References [1,2,9,10] predicated on model reference adaptive control. The work of Narendra and Annaswamy [3] and Hong *et al.* [11] on *linear* direct adaptive control is most closely related to the results presented herein. Specifically, specializing our result to single-input linear systems with no internal dynamics and constant disturbances, we recover the result given in Reference [11].

The contents of the paper are as follows. In Section 2 we present our main direct adaptive control framework for adaptive stabilization, disturbance rejection, and command following of multivariable non-linear uncertain systems with matched exogenous bounded disturbances. In Section 3 we extend the results of Section 2 to non-linear uncertain systems with exogenous L_2 disturbances without a matching condition requirement. Several illustrative numerical examples are presented in Section 4 to demonstrate the efficacy of the proposed direct adaptive stabilization and tracking framework. Finally, in Section 5 we draw some conclusions.

2. ADAPTIVE CONTROL FOR NON-LINEAR SYSTEMS WITH EXOGENOUS DISTURBANCES

In this section we begin by considering the problem of characterizing adaptive feedback control laws for non-linear uncertain systems with exogenous disturbances. Specifically, consider the following controlled non-linear uncertain system \mathcal{G} given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t) + J(x(t))w(t), \quad x(0) = x_0, \quad t \geq 0 \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $w(t) \in \mathbb{R}^d$, $t \geq 0$, is a known bounded disturbance vector, $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and satisfies $f(0) = 0$, $G: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$, and $J: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ is a disturbance weighting matrix function with *unknown* entries. The control input $u(\cdot)$ in (1) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$. Furthermore, for the non-linear system \mathcal{G} we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $f(\cdot)$, $G(\cdot)$, $J(\cdot)$, $u(\cdot)$ and $w(\cdot)$ satisfy sufficient regularity conditions such that (1) has a unique solution forward in time. For the statement of the following result recall the definition of zero-state observability given in Reference [12].

Theorem 2.1.

Consider the non-linear system \mathcal{G} given by (1). Assume there exists a matrix $K_g \in \mathbb{R}^{m \times s}$ and functions $\hat{G}: \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ and $F: \mathbb{R}^n \rightarrow \mathbb{R}^s$, with $F(0) = 0$, such that the zero solution $x(t) \equiv 0$ to

$$\dot{x}(t) = f(x(t)) + G(x(t))\hat{G}(x(t))K_g F(x(t)) \triangleq f_c(x(t)), \quad x(0) = x_0, \quad t \geq 0 \quad (2)$$

is globally asymptotically stable. Furthermore, assume there exists a matrix $\Psi \in \mathbb{R}^{m \times d}$ and a function $\hat{J}: \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ such that $G(x)\hat{J}(x)\Psi = J(x)$. In addition, assume that \mathcal{G} is zero-state observable with $w(t) \equiv 0$ and output $y \triangleq \ell(x)$, where $\ell: \mathbb{R}^n \rightarrow \mathbb{R}^l$, and let $V_s: \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $V_s(\cdot)$ is continuously differentiable, positive definite, radially unbounded, $V_s(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 = V'_s(x)f_c(x) + \ell^T(x)\ell(x) \quad (3)$$

Finally, let $Q_1 \in \mathbb{R}^{m \times m}$, $Q_2 \in \mathbb{R}^{m \times m}$, $Y \in \mathbb{R}^{s \times s}$, and $Z \in \mathbb{R}^{d \times d}$ be positive definite. Then the adaptive feedback control law

$$u(t) = \hat{G}(x(t))K(t)F(x(t)) + \hat{J}(x(t))\Phi(t)w(t) \quad (4)$$

where $K(t) \in \mathbb{R}^{m \times s}$, $t \geq 0$, and $\Phi(t) \in \mathbb{R}^{m \times d}$, $t \geq 0$, with update laws

$$\dot{K}(t) = -\frac{1}{2}Q_1\hat{G}^T(x(t))G^T(x(t))V'_s{}^T(x(t))F^T(x(t))Y \quad (5)$$

$$\dot{\Phi}(t) = -\frac{1}{2}Q_2\hat{J}^T(x(t))G^T(x(t))V'_s{}^T(x(t))w^T(t)Z \quad (6)$$

guarantees that the solution $(x(t), K(t), \Phi(t)) \equiv (0, K_g, -\Psi)$ of the closed-loop system given by (1), (4), (5), and (6) is Lyapunov stable and $\ell(x(t)) \rightarrow 0$ as $t \rightarrow \infty$. If, in addition, $\ell^T(x)\ell(x) > 0$, $x \neq 0$, then $x(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $x_0 \in \mathbb{R}^n$.

Proof. Note that with $u(t)$, $t \geq 0$, given by (4) it follows from (1) that

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + G(x(t))\hat{G}(x(t))K(t)F(x(t)) + G(x(t))\hat{J}(x(t))\Phi(t)w(t) + J(x(t))w(t), \\ x(0) &= x_0, \quad t \geq 0\end{aligned}\tag{7}$$

or, equivalently, using the fact that $G(x)\hat{J}(x)\Psi = J(x)$,

$$\begin{aligned}\dot{x}(t) &= f_c(x(t)) + G(x(t))\hat{G}(x(t))(K(t) - K_g)F(x(t)) + G(x(t))\hat{J}(x(t))(\Phi(t) + \Psi)w(t), \\ x(0) &= x_0, \quad t \geq 0\end{aligned}\tag{8}$$

To show Lyapunov stability of the closed-loop system (5), (6) and (8) consider the Lyapunov function candidate

$$V(x, K, \Phi) = V_s(x) + \text{tr } Q_1^{-1}(K - K_g)Y^{-1}(K - K_g)^T + \text{tr } Q_2^{-1}(\Phi + \Psi)Z^{-1}(\Phi + \Psi)^T\tag{9}$$

Note that global asymptotic stability of the zero solution $x(t) \equiv 0$ to (2) and zero-state observability of (1) with $w(t) \equiv 0$ and output $\ell(x)$, guarantees the existence of a continuously differentiable, positive-definite, radially unbounded function $V_s: \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying (3). Furthermore, note that $V(0, K_g, -\Psi) = 0$ and, since $V_s(\cdot)$, Q_1 , Q_2 , Y and Z are positive definite, $V(x, K, \Phi) > 0$ for all $(x, K, \Phi) \neq (0, K_g, -\Psi)$. In addition, $V(x, K, \Phi)$ is radially unbounded. Now, letting $x(t)$, $t \geq 0$, denote the solution to (8) and using (3), (5) and (6), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}\dot{V}(x(t), K(t), \Phi(t)) &= V'_s(x(t))[f_c(x(t)) + G(x(t))\hat{G}(x(t))(K(t) - K_g)F(x(t)) \\ &\quad + G(x(t))\hat{J}(x(t))(\Phi(t) + \Psi)w(t)] + 2\text{tr } Q_1^{-1}(K(t) - K_g)Y^{-1}\dot{K}^T(t) \\ &\quad + 2\text{tr } Q_2^{-1}(\Phi(t) + \Psi)Z^{-1}\dot{\Phi}^T(t) \\ &= -\ell^T(x(t))\ell(x(t)) + \text{tr}[(K(t) - K_g)F(x(t))V'_s(x(t))G(x(t))\hat{G}(x(t))] \\ &\quad + \text{tr}[(\Phi(t) + \Psi)w(t)V'_s(x(t))G(x(t))\hat{J}(x(t))] \\ &\quad - \text{tr}[(K(t) - K_g)F(x(t))V'_s(x(t))G(x(t))\hat{G}(x(t))] \\ &\quad - \text{tr}[(\Phi(t) + \Psi)w(t)V'_s(x(t))G(x(t))\hat{J}(x(t))] \\ &= -\ell^T(x(t))\ell(x(t)) \\ &\leq 0\end{aligned}\tag{10}$$

which proves that the solution $(x(t), K(t), \Phi(t)) \equiv (0, K_g, -\Psi)$ to (5), (6), and (8) is Lyapunov stable. Furthermore, it follows from Theorem 4.4 of Reference [9] that $\ell(x(t)) \rightarrow 0$ as $t \rightarrow \infty$. Finally, if $\ell^T(x)\ell(x) > 0$, $x \neq 0$, then $x(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $x_0 \in \mathbb{R}^n$. \square

Remark 2.1

Note that the conditions in Theorem 2.1 imply that $x(t) \rightarrow 0$ as $t \rightarrow \infty$ and hence it follows from (5) and (6) that $(x(t), K(t), \Phi(t)) \rightarrow \mathcal{M} \triangleq \{(x, K, \Phi) \in \mathbb{R}^n \times \mathbb{R}^{m \times s} \times \mathbb{R}^{m \times d} : x = 0, \dot{K} = 0, \dot{\Phi} = 0\}$ as $t \rightarrow \infty$.

Remark 2.2.

Theorem 2.1 is also valid for non-linear *time-varying* uncertain systems \mathcal{G}_t of the form

$$\dot{x}(t) = f(t, x(t)) + G(t, x(t))u(t) + J(t, x(t))w(t), \quad x(0) = x_0, \quad t \geq 0 \quad (11)$$

where $f: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and satisfies $f(t, 0) = 0$, $t \geq 0$, $G: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$, and $J: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$. In particular, replacing $F: \mathbb{R}^n \rightarrow \mathbb{R}^s$ by $F: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^s$, where $F(t, 0) = 0$, $t \geq 0$, $\hat{G}: \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ by $\hat{G}: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$, and requiring $G(t, x)\hat{J}(t, x)\Psi = J(t, x)$, where $\hat{J}: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ and $t \geq 0$, in place of $G(x)\hat{J}(x)\Psi = J(x)$, it follows by using identical arguments as in the proof of Theorem 2.1 that the adaptive feedback control law

$$u(t) = \hat{G}(t, x(t))K(t)F(t, x(t)) + \hat{J}(t, x(t))\Phi(t)w(t), \quad (12)$$

with the update laws

$$\dot{K}(t) = -\frac{1}{2}Q_1 \hat{G}^T(t, x(t))G^T(t, x(t))V_s^T(x(t))F^T(t, x(t))Y \quad (13)$$

$$\dot{\Phi}(t) = -\frac{1}{2}Q_2 \hat{J}^T(t, x(t))G^T(t, x(t))V_s^T(x(t))w^T(t)Z \quad (14)$$

where $V_s'(x)$ satisfies (3) with $f_c(x) = f(t, x) + G(t, x)\hat{G}(t, x)K_g F(t, x)$, guarantees that the solution $(x(t), K(t), \Phi(t)) \equiv (0, K_g, -\Psi)$ of the closed-loop system (11)–(14) is Lyapunov stable and $x(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $x_0 \in \mathbb{R}^n$.

Remark 2.3

It follows from Remark 2.2 that Theorem 2.1 can also be used to construct adaptive tracking controllers for non-linear uncertain systems. Specifically, let $r_d(t) \in \mathbb{R}^n$, $t \geq 0$, denote a command input and define the error state $e(t) \triangleq x(t) - r_d(t)$. In this case, the error dynamics are given by

$$\dot{e}(t) = f_i(t, e(t)) + G(t, e(t))u(t) + J_i(t, e(t))w_i(t), \quad e(0) = e_0, \quad t \geq 0 \quad (15)$$

where $f_i(t, e(t)) = f(e(t) + r_d(t)) - n(t)$, with $f(r_d(t)) = n(t)$, and $J_i(t, e(t))w_i(t) = n(t) - \dot{r}_d(t) + J(t, e(t))w(t)$. Now, the adaptive tracking control law (12)–(14), with $x(t)$ replaced by $e(t)$, guarantees that $e(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $e_0 \in \mathbb{R}^n$.

It is important to note that the adaptive control law (4)–(6) does *not* require explicit knowledge of the gain matrix K_g , the disturbance matching matrix Ψ , and the disturbance weighting matrix

function $J(x)$; even though Theorem 2.1 requires the existence of $K_g, F(x), \hat{G}(x), \hat{J}(x)$, and Ψ such that the zero solution $x(t) \equiv 0$ to (2) is globally asymptotically stable and the matching condition $G(x)\hat{J}(x)\Psi = J(x)$ holds. Furthermore, no specific structure on the non-linear dynamics $f(x)$ is required to apply Theorem 2.1; all that is required is the existence of $F(x)$ such that the zero solution $x(t) \equiv 0$ to (2) is asymptotically stable so that (3) holds. However, if (1) is in normal form with asymptotically stable internal dynamics [7], then we can always construct a function $F: \mathbb{R}^n \rightarrow \mathbb{R}^s$, with $F(0) = 0$, such that the zero solution $x(t) \equiv 0$ to (2) is globally asymptotically stable *without* requiring knowledge of the system dynamics. These facts are exploited below to construct nonlinear adaptive feedback controllers for non-linear uncertain systems. For simplicity of exposition in the ensuing discussion we assume that $J(x) = D$, where $D \in \mathbb{R}^{n \times d}$ is a disturbance weighting matrix with unknown entries.

To elucidate the above discussion assume that the non-linear uncertain system \mathcal{G} is generated by

$$q_i^{(r_i)}(t) = f_{u_i}(q(t)) + \sum_{j=1}^m G_{s(i,j)}(q(t))u_j(t) + \sum_{k=1}^d \hat{D}_{(i,k)}w_k(t), \quad q(0) = q_0, \quad t \geq 0, \quad i = 1, \dots, m \quad (16)$$

where $q_i^{(r_i)}$ denotes the r_i th derivative of q_i , r_i denotes the relative degree with respect to the output q_i , $f_{u_i}(q) = f_{u_i}(q_1, \dots, q_1^{(r_1-1)}, \dots, q_m, \dots, q_m^{(r_m-1)})$, $G_{s(i,j)}(q) = G_{s(i,j)}(q_1, \dots, q_1^{(r_1-1)}, \dots, q_m, \dots, q_m^{(r_m-1)})$, $\hat{D}_{(i,k)} \in \mathbb{R}$, $i = 1, \dots, m$, $k = 1, \dots, d$, and $w_k(t) \in \mathbb{R}$, $t \geq 0$, $k = 1, \dots, d$. Here, we assume that the square matrix function $G_s(q)$ composed of the entries $G_{s(i,j)}(q)$, $i, j = 1, \dots, m$, is such that $\det G_s(q) \neq 0$, $q \in \mathbb{R}^{\hat{r}}$, where $\hat{r} = r_1 + \dots + r_m$ is the (vector) relative degree of (16). Furthermore, since (16) is in a form where it does not possess internal dynamics, it follows that $\hat{r} = n$. The case where (16) possesses internal dynamics is discussed below.

Next, define $x_i \triangleq [q_i, \dots, q_i^{(r_i-2)}]^T$, $i = 1, \dots, m$, $x_{m+1} \triangleq [q_1^{(r_1-1)}, \dots, q_m^{(r_m-1)}]^T$, and $x \triangleq [x_1^T, \dots, x_{m+1}^T]^T$, so that (16) can be described by (1) with

$$f(x) = \tilde{A}x + \tilde{f}_u(x), \quad G(x) = \begin{bmatrix} 0_{(n-m) \times m} \\ G_s(x) \end{bmatrix}, \quad J(x) = D = \begin{bmatrix} 0_{(n-m) \times d} \\ \hat{D} \end{bmatrix} \quad (17)$$

where

$$\tilde{A} = \begin{bmatrix} A_0 \\ 0_{m \times n} \end{bmatrix}, \quad \tilde{f}_u(x) = \begin{bmatrix} 0_{(n-m) \times 1} \\ f_u(x) \end{bmatrix},$$

$A_0 \in \mathbb{R}^{(n-m) \times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [13], $f_u: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is an unknown function and satisfies $f_u(0) = 0$, $G_s: \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$, and $\hat{D} \in \mathbb{R}^{m \times d}$. Here, we assume that $f_u(x)$ is unknown and is parameterized as $f_u(x) = \Theta f_n(x)$, where $f_n: \mathbb{R}^n \rightarrow \mathbb{R}^q$ and satisfies $f_n(0) = 0$, and $\Theta \in \mathbb{R}^{m \times q}$ is a matrix of uncertain constant parameters. Note that $\hat{J}(x)$ and Ψ in Theorem 2.1 can be taken as $\hat{J}(x) = G_s^{-1}(x)$ and $\Psi = \hat{D}$ so that $G(x)\hat{J}(x)\Psi = J(x) = D$ is satisfied.

Next, to apply Theorem 2.1 to the uncertain system (1) with $f(x)$, $G(x)$, and $J(x)$ given by (17), let $K_g \in \mathbb{R}^{m \times s}$, where $s = q + r$, be given by

$$K_g = [\Theta_n - \Theta, \Phi_n], \quad (18)$$

where $\Theta_n \in \mathbb{R}^{m \times q}$ and $\Phi_n \in \mathbb{R}^{m \times r}$ are known matrices, and let

$$F(x) = \begin{bmatrix} f_n(x) \\ \hat{f}_n(x) \end{bmatrix} \tag{19}$$

where $\hat{f}_n: \mathbb{R}^n \rightarrow \mathbb{R}^r$ and satisfies $\hat{f}_n(0) = 0$ is an arbitrary function. In this case, it follows that, with $\hat{G}(x) = G_s^{-1}(x)$,

$$\begin{aligned} f_c(x) &= f(x) + G(x)\hat{G}(x)K_g F(x) \\ &= \tilde{A}x + \tilde{f}_u(x) + \begin{bmatrix} 0_{(n-m) \times m} \\ G_s(x) \end{bmatrix} G_s^{-1}(x) [\Theta_n f_n(x) - \Theta f_n(x) + \Phi_n \hat{f}_n(x)] \\ &= \tilde{A}x + \begin{bmatrix} 0_{(n-m) \times 1} \\ \Theta_n f_n(x) + \Phi_n \hat{f}_n(x) \end{bmatrix} \end{aligned} \tag{20}$$

Now, since $\Theta_n \in \mathbb{R}^{m \times q}$ and $\Phi_n \in \mathbb{R}^{m \times r}$ are arbitrary constant matrices and $\hat{f}_n: \mathbb{R}^n \rightarrow \mathbb{R}^r$ is an arbitrary function we can always construct K_g and $F(x)$ without knowledge of $f(x)$ such that the zero solution $x(t) \equiv 0$ to (2) can be made globally asymptotically stable. In particular, choosing $\Theta_n f_n(x) + \Phi_n \hat{f}_n(x) = \hat{A}x$, where $\hat{A} \in \mathbb{R}^{m \times n}$, it follows that (20) has the form $f_c(x) = A_c x$, where $A_c = [A_0^T, \hat{A}^T]^T$ is in multivariable controllable canonical form. Hence, choosing \hat{A} such that A_c is asymptotically stable, it follows from converse Lyapunov theory that there exists a positive-definite matrix P satisfying the Lyapunov equation

$$0 = A_c^T P + P A_c + R \tag{21}$$

where R is positive definite. In this case, with Lyapunov function $V_s(x) = x^T P x$, the adaptive feedback controller (4) with update laws (5), (6), or, equivalently,

$$\dot{K}(t) = -Q_1 \hat{G}^T(x(t)) G^T(x(t)) P x(t) F^T(x(t)) Y \tag{22}$$

$$\dot{\Phi}(t) = -Q_2 \hat{J}^T(x(t)) G^T(x(t)) P x(t) w^T(t) Z \tag{23}$$

guarantees global asymptotic stability of the *non-linear* uncertain dynamical system (1) where $f(x)$, $G(x)$ and $J(x)$ are given by (17). As mentioned above, it is important to note that it is not necessary to utilize a feedback linearizing function $F(x)$ to produce a linear $f_c(x)$. However, when the system is in normal form, a feedback linearizing function $F(x)$ provides considerable simplification in constructing $V'_s(x)$ necessary in computing the update laws (5) and (6).

A similar construction as discussed above can be used in the case where (1) is in normal form with input-to-state stable zero dynamics [8] and $w(t) = 0$. In this case, (16) is given by

$$q_i^{(r_i)}(t) = f_{u_i}(q(t), z(t)) + \sum_{j=1}^m G_{s(i,j)}(q(t), z(t)) u_j(t), \quad q(0) = q_0, \quad t \geq 0, \quad i = 1, \dots, m \tag{24}$$

$$\dot{z}(t) = f_z(q(t), z(t)), \quad z(0) = z_0 \tag{25}$$

where $f_z: \mathbb{R}^{\hat{r}} \times \mathbb{R}^{n-\hat{r}} \rightarrow \mathbb{R}^{n-\hat{r}}$, $\hat{r} < n$, and where we have assumed for simplicity of exposition that the distribution spanned by the vector fields $\text{col}_1(G(x)), \dots, \text{col}_m(G(x))$, where $\text{col}_i(G(x))$ denotes the i th column of $G(x)$, is involutive [7]. Here, we assume that the zero solution $z(t) \equiv 0$ to (25) is input-to-state stable with q viewed as the input. Next, define $x \triangleq [\hat{x}^T, z^T]^T$, where $\hat{x} \triangleq [x_1^T, \dots, x_{m+1}^T]^T \in \mathbb{R}^{\hat{r}}$. Now, since the zero solution $\hat{x}(t) \equiv 0$ can be made asymptotically stable by a similar construction as discussed above and since the zero dynamics given by (25) are input-to-state stable, it follows from Lemma 5.6 of Reference [9] that the zero solution $x(t) \equiv 0$ to (1) with $w(t) \equiv 0$ is globally asymptotically stable.

Next, we consider the case where $f(x)$ and $G(x)$ are uncertain and $\hat{r} = n$. Specifically, we assume that $G_s(x)$ is unknown and is parameterized as $G_s(x) = B_s G_n(x)$, where $G_n: \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ is known and satisfies $\det G_n(x) \neq 0$, $x \in \mathbb{R}^n$, and $B_s \in \mathbb{R}^{m \times m}$, with $\det B_s \neq 0$, is an unknown symmetric sign definite matrix but the sign definiteness of B_s is known; that is, $B_s > 0$ or $B_s < 0$. For the statement of the next result define $B_0 \triangleq [0_{m \times (n-m)}, I_m]^T$ for $B_s > 0$, and $B_0 \triangleq [0_{m \times (n-m)}, -I_m]^T$ for $B_s < 0$.

Corollary 2.1.

Consider the non-linear system \mathcal{G} given by (1) with $f(x)$, $G(x)$, and $J(x)$ given by (17) and $G_s(x) = B_s G_n(x)$, where B_s is an unknown symmetric matrix and the sign definiteness of B_s is known. Assume there exists a matrix $K_g \in \mathbb{R}^{m \times s}$ and a function $F: \mathbb{R}^n \rightarrow \mathbb{R}^s$, with $F(0) = 0$, such that the zero solution $x(t) \equiv 0$ to (2) is globally asymptotically stable. Furthermore, assume that \mathcal{G} is zero-state observable with $w(t) \equiv 0$ and output $y \triangleq \ell(x)$, where $\ell: \mathbb{R}^n \rightarrow \mathbb{R}^t$, and let $V_s: \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $V_s(\cdot)$ is continuously differentiable, positive definite, radially unbounded, $V_s(0) = 0$, and (3) holds. Finally, let $Y \in \mathbb{R}^{s \times s}$ and $Z \in \mathbb{R}^{d \times d}$ be positive definite. Then the adaptive feedback control law

$$u(t) = G_n^{-1}(x(t))K(t)F(x(t)) + G_n^{-1}(x(t))\Phi(t)w(t) \quad (26)$$

where $K(t) \in \mathbb{R}^{m \times s}$, $t \geq 0$, and $\Phi(t) \in \mathbb{R}^{m \times d}$, $t \geq 0$, with update laws

$$\dot{K}(t) = -\frac{1}{2}B_0^T V_s^T(x(t))F^T(x(t))Y \quad (27)$$

$$\dot{\Phi}(t) = -\frac{1}{2}B_0^T V_s^T(x(t))w^T(t)Z \quad (28)$$

guarantees that the solution $(x(t), K(t), \Phi(t)) \equiv (0, K_g, -\Psi)$, where $\Psi \in \mathbb{R}^{m \times d}$, of the closed-loop system given by (1), (26)–(28) is Lyapunov stable and $\ell(x(t)) \rightarrow 0$ as $t \rightarrow \infty$. If, in addition, $\ell^T(x)\ell(x) > 0$, $x \neq 0$, then $x(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $x_0 \in \mathbb{R}^n$.

Proof. The result is a direct consequence of Theorem 2.1. First, let $\hat{G}(x) = \hat{J}(x) = G_n^{-1}(x)$ and $\Psi = B_s^{-1}\hat{D}$ so that $G(x)\hat{J}(x)\Psi = D$ and let $K_g = B_s^{-1}[\Theta_n - \Theta, \Phi_n]$. Next, since Q_1 and Q_2 are arbitrary positive-definite matrices, Q_1 in (5) and Q_2 in (6) can be replaced by $q_1|B_s|^{-1}$ and $q_2|B_s|^{-1}$, respectively, where q_1, q_2 are positive constants and $|B_s| = (B_s^2)^{1/2}$, where $(\cdot)^{1/2}$ denotes the (unique) positive-definite square root. Now, since B_s is symmetric and sign definite it follows from the Schur decomposition that $B_s = UD_B U^T$, where U is orthogonal and D_B is real diagonal. Hence, $|B_s|^{-1}B^T = [0_{m \times (n-m)}, \mathcal{J}_m] = B_0^T$, where $\mathcal{J}_m = I_m$ for $B_s > 0$ and $\mathcal{J}_m = -I_m$ for $B_s < 0$. Now, (5) and (6), with $q_1 Y$ and $q_2 Z$ replaced by Y and Z , imply (27) and (28), respectively. \square

It is important to note that if, as discussed above K_g , and $F(x)$ are constructed to give $f_c(x) = A_c x$ in (2), where A_c is an asymptotically stable matrix in multivariable controllable canonical form, then considerable simplification occurs in Corollary 2.1. Specifically, in this case $V_s(x) = x^T P x$, where $P > 0$ satisfies (21), and hence (27), (28) become

$$\dot{K}(t) = -B_0^T P x(t) F^T(x(t)) Y \quad (29)$$

$$\dot{\Phi}(t) = -B_0^T P x(t) w^T(t) Z \quad (30)$$

Finally, we note that by setting $m = d = 1$, $s = n$, $w(t) \equiv 1$, $F(x) = x$, $f(x) = Ax$, where $A = [A_0^T, \theta^T]^T$, $A_0 \in \mathbb{R}^{(n-1) \times n}$ is a known matrix, and $\theta \in \mathbb{R}^{1 \times n}$ is an unknown vector, $G(x) = [0_{1 \times (n-1)}, b]^T$, where $b \neq 0$ is unknown but $\text{sign } b \triangleq b/|b|$ is known, and $J(x) = [0_{1 \times (n-1)}, \hat{d}]^T$, Corollary 2.1 specializes to the results given in Reference [11].

3. ADAPTIVE CONTROL FOR NON-LINEAR SYSTEMS WITH L_2 DISTURBANCES

In this section we consider the problem of characterizing adaptive feedback control laws for non-linear uncertain systems with exogenous L_2 disturbances. Specifically, we consider the following controlled non-linear uncertain system \mathcal{G} given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t) + J(x(t))w(t), \quad x(0) = x_0, \quad w(\cdot) \in L_2, \quad t \geq 0 \quad (31)$$

with performance variables

$$z(t) = h(x(t)) \quad (32)$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $w(t) \in \mathbb{R}^d$, $t \geq 0$, is an unknown bounded energy L_2 disturbance, $z(t) \in \mathbb{R}^p$, $t \geq 0$, is a performance variable, $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and satisfies $f(0) = 0$, $G: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$, $J: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ and $h: \mathbb{R}^n \rightarrow \mathbb{R}^p$ and satisfies $h(0) = 0$. The following theorem generalizes Theorem 2.1 to non-linear uncertain systems with exogenous L_2 disturbances.

Theorem 3.1

Consider the non-linear system \mathcal{G} given by (31) and (32). Assume there exists a matrix $K_g \in \mathbb{R}^{m \times s}$ and functions $\hat{G}: \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ and $F: \mathbb{R}^n \rightarrow \mathbb{R}^s$, with $F(0) = 0$, such that the zero solution $x(t) \equiv 0$ to (2) is globally asymptotically stable. Furthermore, assume there exists a continuously differentiable function $V_s: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $V_s(\cdot)$ is positive definite, radially unbounded, $V_s(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 = V'_s(x) f_c(x) + \Gamma(x) \quad (33)$$

where

$$\Gamma(x) \triangleq \frac{1}{4\gamma^2} V'_s(x) J(x) J^T(x) V_s^T(x) + h^T(x) h(x) \quad (34)$$

Finally, let $Q \in \mathbb{R}^{m \times m}$ and $Y \in \mathbb{R}^{s \times s}$ be positive definite. Then the adaptive feedback control law

$$u(t) = \hat{G}(x(t))K(t)F(x(t)) \quad (35)$$

where $K(t) \in \mathbb{R}^{m \times s}$, $t \geq 0$, with update law

$$\dot{K}(t) = -\frac{1}{2}Q\hat{G}^T(x(t))G^T(x(t))V_s'^T(x(t))F^T(x(t))Y \quad (36)$$

guarantees that the solution $(x(t), K(t)) \equiv (0, K_g)$ of the undisturbed ($w(t) \equiv 0$) closed-loop system given by (31), (35) and (36) is Lyapunov stable and $h(x(t)) \rightarrow 0$ as $t \rightarrow \infty$. If, in addition, $h^T(x)h(x) > 0$, $x \neq 0$, then $x(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $x_0 \in \mathbb{R}^n$. Furthermore, the solution $x(t)$, $t \geq 0$, to the closed-loop system given by (31), (35) and (36) satisfies the non-expansivity constraint

$$\int_0^T z^T(t)z(t) dt \leq \gamma^2 \int_0^T w^T(t)w(t) dt + V(x(0), K(0)), \quad T \geq 0, \quad w(\cdot) \in L_2 \quad (37)$$

where

$$V(x, K) \triangleq V_s(x) + \text{tr} Q^{-1}(K - K_g)Y^{-1}(K - K_g)^T \quad (38)$$

Proof. Note that with $u(t)$, $t \geq 0$, given by (35) it follows from (31) that

$$\dot{x}(t) = f(x(t)) + G(x(t))\hat{G}(x(t))K(t)F(x(t)) + J(x(t))w(t), \quad x(0) = x_0, \quad w(\cdot) \in L_2, \quad t \geq 0 \quad (39)$$

or, equivalently, using the definition for $f_c(x)$ given in (2),

$$\dot{x}(t) = f_c(x(t)) + G(x(t))\hat{G}(x(t))(K(t) - K_g)F(x(t)) + J(x(t))w(t), \quad x(0) = x_0, \quad w(\cdot) \in L_2, \quad t \geq 0 \quad (40)$$

To show Lyapunov stability of the closed-loop system (36) and (40) consider the Lyapunov function candidate given by (38). Note that $V(0, K_g) = 0$ and, since $V_s(\cdot)$, Q , and Y are positive definite, $V(x, K) > 0$ for all $(x, K) \neq (0, K_g)$. Furthermore, $V(x, K)$ is radially unbounded. Now, Lyapunov stability of the undisturbed ($w(t) \equiv 0$) closed-loop system (36) and (40) as well as $x(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $x_0 \in \mathbb{R}^n$ follows as in the proof of Theorem 2.1. To show that the non-expansivity constraint (37) holds, note that, for all $w \in \mathbb{R}^d$,

$$\begin{aligned} 0 &\leq \left[\frac{1}{2\gamma} J^T(x) V_s'^T(x) - \gamma w \right]^T \left[\frac{1}{2\gamma} J^T(x) V_s'^T(x) - \gamma w \right] \\ &= \Gamma(x) + \gamma^2 w^T w - z^T z - V_s'^T(x) J(x) w \end{aligned} \quad (41)$$

Now, let $w(\cdot) \in L_2$ and let $x(t)$, $t \geq 0$, denote the solution of the closed-loop systems (36) and (40). Then the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned} \dot{V}(x(t), K(t)) &= V'_s(x(t)) [f_c(x(t)) + G(x(t))\hat{G}(x(t))(K(t) - K_g)F(x(t)) + J(x(t))w(t)] \\ &\quad + 2\text{tr} Q^{-1}(K(t) - K_g)Y^{-1}\dot{K}^T(t) \\ &= -\Gamma(x(t)) + \text{tr} [(K(t) - K_g)F(x(t))V'_s(x(t))G(x(t))\hat{G}(x(t))] \\ &\quad + V'_s(x(t))J(x(t))w(t) - \text{tr} [(K(t) - K_g)F(x(t))V'_s(x(t))G(x(t))\hat{G}(x(t))] \\ &= -\Gamma(x(t)) + V'_s(x(t))J(x(t))w(t) \\ &\leq \gamma^2 w^T(t)w(t) - z^T(t)z(t) \end{aligned} \tag{42}$$

Now, integrating (42) over $[0, T]$ yields

$$V(x(T), K(T)) \leq \int_0^T [\gamma^2 w^T(t)w(t) - z^T(t)z(t)] dt + V(x(0), K(0)), \quad T \geq 0, \quad w(\cdot) \in L_2 \tag{43}$$

which, by noting that $V(x(T), K(T)) \geq 0$, $T \geq 0$, yields (37). □

It is important to note that unlike Theorem 2.1 requiring a matching condition on the disturbance, Theorem 3.1 does not require any such matching condition. Furthermore, as shown in Section 2, if (31) is in normal form with asymptotically stable internal dynamics, then we can always construct a function $F: \mathbb{R}^n \rightarrow \mathbb{R}^s$, with $F(0) = 0$, such that the zero solution to (2) to globally asymptotically stable without requiring knowledge of the system dynamics. In addition, in the case where $J(x) = D$ and $h(x) = Ex$, the adaptive controller (36) can be constructed to guarantee the non-expansivity constraint (37) using standard *linear* H_∞ methods. Specifically, choosing $f_c(x) = A_c x$, where A_c is asymptotically stable and in multivariable controllable canonical form, it follows from standard H_∞ theory [14] that if (A_c, E) is observable, $\|G(s)\|_\infty < \gamma$, where $G(s) = E(sI_n - A_c)^{-1}D$, if and only if there exists a positive-definite matrix P satisfying the bounded real Riccati equation

$$0 = A_c^T P + P A_c + \gamma^{-2} P D D^T P + E^T E \tag{44}$$

It is well known that (44) has a non-negative-definite solution if and only if the Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A_c & \gamma^{-2} D D^T \\ -E^T E & -A_c^T \end{bmatrix} \tag{45}$$

has no purely imaginary eigenvalues. If, in addition, $E^T E > 0$, then $P > 0$. In this case, with Lyapunov function $V_s(x) = x^T P x$, the adaptive feedback controller (35) with update law (36), or, equivalently,

$$\dot{K}(t) = -Q \hat{G}^T(x(t)) G^T(x(t)) P x(t) F^T(x(t)) Y \tag{46}$$

guarantees global asymptotic stability of the non-linear undisturbed ($w(t) \equiv 0$) dynamical system (31), where $f(x)$ and $G(x)$ are given by (17). Furthermore, the solution $x(t)$, $t \geq 0$, of the closed-loop *non-linear* dynamical system (31) is guaranteed to satisfy the non-expansivity constraint (37).

Finally, if $f(x)$ and $G(x)$ given by (17) are uncertain and $G_s(x) = B_s G_n(x)$, where the sign definiteness of B_s is known, then using an identical approach as in Section 2, it can be shown that the adaptive feedback control law

$$u(t) = G_n^{-1}(x(t))K(t)F(x(t)) \quad (47)$$

with update law

$$\dot{K}(t) = -\frac{1}{2}B_0^T V_s'^T(x(t))F^T(x(t))Y \quad (48)$$

where B_0 is defined as in Section 2, guarantees asymptotic stability and non-expansivity of (31).

4. ILLUSTRATIVE NUMERICAL EXAMPLES

In this section we present several numerical examples to demonstrate the utility of the proposed direct adaptive control framework for adaptive stabilization, disturbance rejection, and command following.

Example 4.1

Consider the uncertain controlled Van der Pol oscillator given by

$$\ddot{z}(t) - \varepsilon(\alpha - z^2(t))\dot{z}(t) + \beta z(t) = bu(t), \quad z(0) = z_0, \quad \dot{z}(0) = \dot{z}_0, \quad t \geq 0 \quad (49)$$

where $\alpha, \beta, \varepsilon, b \in \mathbb{R}$ are unknown. Note that with $x_1 = z$ and $x_2 = \dot{z}$, (49) can be written in state-space form (1) with $x = [x_1, x_2]^T$, $f(x) = [x_2, -\beta x_1 + \varepsilon(\alpha - x_1^2)x_2]^T$, and $G(x) = [0, b]^T$. Here, we assume that $f(x)$ is unknown and can be parameterized as $f(x) = [x_2, \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^2 x_2]^T$, where θ_1, θ_2 , and θ_3 are unknown constants. Furthermore, we assume that sign b is known. Next, let $G_n(x) = 1$, $F(x) = [x_1, x_2, x_1^2 x_2]^T$, and $K_g = 1/b[\theta_{n_1} - \theta_1, \theta_{n_2} - \theta_2, -\theta_3]$, where $\theta_{n_1}, \theta_{n_2}$ are arbitrary scalars, so that

$$\begin{aligned} f_c(x) &= f(x) + \begin{bmatrix} 0 \\ b \end{bmatrix} \frac{1}{b} [\theta_{n_1} - \theta_1, \theta_{n_2} - \theta_2, -\theta_3] F(x) \\ &= \begin{bmatrix} 0 & 1 \\ \theta_{n_1} & \theta_{n_2} \end{bmatrix} x \end{aligned} \quad (50)$$

Now, with the proper choice of θ_{n_1} and θ_{n_2} , it follows from Corollary 2.1 that the adaptive feedback controller (26) with $w(t) \equiv 0$ guarantees that $x(t) \rightarrow 0$ as $t \rightarrow \infty$. Specifically, here we

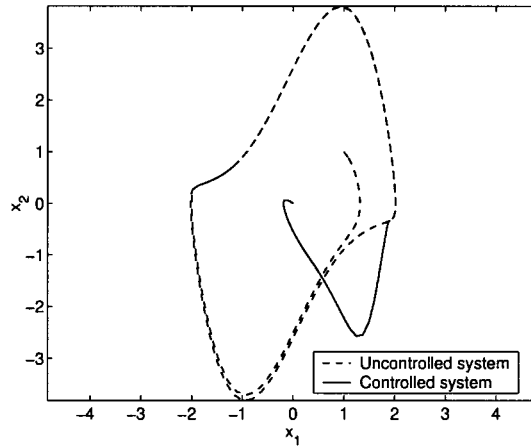


Figure 1. Phase portrait of controlled and uncontrolled Van der Pol oscillator.

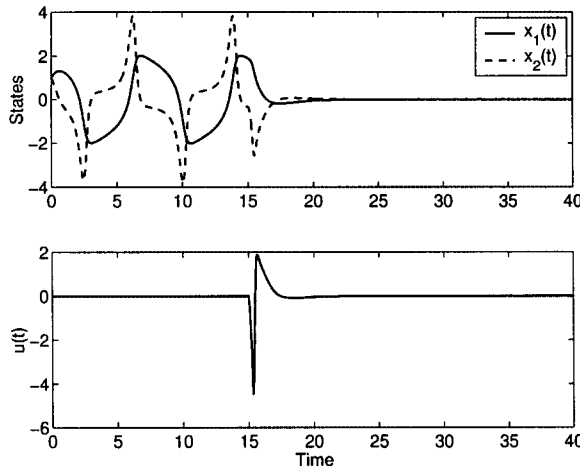


Figure 2. State trajectories and control signal versus time.

choose $\theta_{n_1} = -1$, $\theta_{n_2} = -2$, and $R = 2I_2$, so that P satisfying (21) is given by

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \tag{51}$$

With $\alpha = 1$, $\beta = 1$, $\varepsilon = 2$, $b = 3$, $Y = I_2$, and initial conditions $x(0) = [1, 1]^T$ and $K(0) = [0, 0, 0]$, Figure 1 shows the phase portrait of the controlled and uncontrolled system. Note that the adaptive controller is switched on at $t = 15$ s. Figure 2 shows the state trajectories versus time and the control signal versus time. Finally, Figure 3 shows the adaptive gain history versus time.

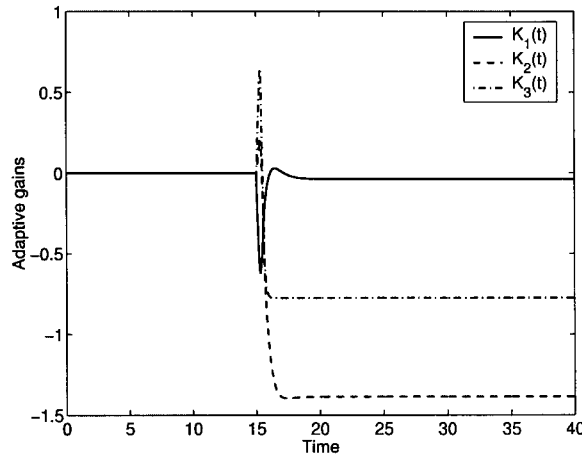


Figure 3. Adaptive gain history versus time.

Example 4.2

The following example considers the utility of the proposed adaptive stabilization framework for systems with time-varying dynamics. Specifically, consider the uncertain controlled Mathieu system given by

$$\ddot{z}(t) + \mu(1 + 2\varepsilon \cos 2t)z(t) = bu(t), \quad z(0) = z_0, \quad \dot{z}(0) = \dot{z}_0, \quad t \geq 0 \quad (52)$$

where $\mu, \varepsilon, b \in \mathbb{R}$ are unknown. Note that with $x_1 = z$ and $x_2 = \dot{z}$, (52) can be written in state-space form (11) with $x = [x_1, x_2]^T$, $f(t, x) = [x_2, -\mu(1 + 2\varepsilon \cos 2t)x_1]^T$, and $G(t, x) = [0, b]^T$. Here, we assume that $\text{sign } b$ is known and $f(t, x)$ can be parameterized as $f(t, x) = [x_2, \theta_1 x_1 + \theta_2 \cos(2t)x_1]^T$, where θ_1 and θ_2 are unknown constants. Next, let $\hat{G}(t, x) = 1$, $F(t, x) = [x_1, \cos(2t)x_1, x_2]^T$, and $K_g = 1/b[\theta_{n_1}, -\theta_1, -\theta_2, \phi_n]$, where θ_{n_1} and ϕ_n are arbitrary scalars, so that

$$f_c(x) = \begin{bmatrix} 0 & 1 \\ \theta_{n_1} & \phi_n \end{bmatrix} x$$

Now, with the proper choice of θ_{n_1} and ϕ_n , it follows from Corollary 2.1 and Remark 2.2 that the adaptive feedback controller (12) with $w(t) \equiv 0$ guarantees that $x(t) \rightarrow 0$ as $t \rightarrow \infty$. Specifically, here we choose $\theta_{n_1} = -1$, $\phi_n = -2$, and $R = 2I_2$, so that P satisfying (21) is given by (51). With $\mu = 1$, $\varepsilon = 0.4$, $b = 3$, $Y = I_3$, and initial conditions $x(0) = [1, 1]^T$ and $K(0) = [0, 0, 0]$, Figure 4 shows the phase portrait of the controlled and uncontrolled system. Note that the adaptive controller is switched on at $t = 15$ s. Figure 5 shows the state trajectories versus time and the control signal versus time. Finally, Figure 6 shows the adaptive gain history versus time.

Example 4.3

The following example considers the utility of the proposed adaptive control framework for command following. Specifically, consider the spring-mass-damper uncertain system with

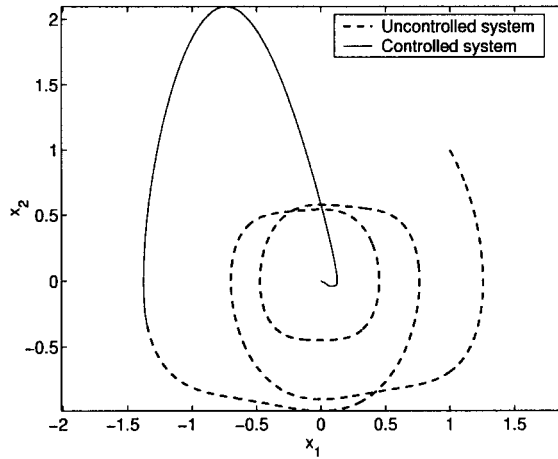


Figure 4. Phase portrait of controlled and uncontrolled Mathieu system.

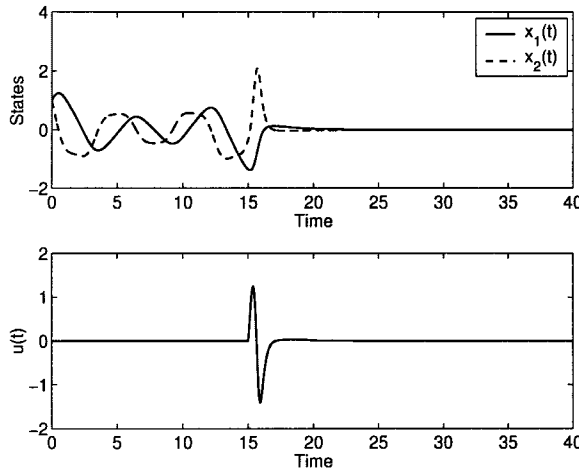


Figure 5. State trajectories and control signal versus time.

nonlinear stiffness given by

$$m\ddot{x}(t) + c\dot{x}(t) + k_1x(t) + k_2x^3(t) = bu(t) + \hat{d}w(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \quad t \geq 0 \quad (53)$$

where $m, c, k_1, k_2 \in \mathbb{R}$ are positive unknown constants, and b is unknown but sign b is known. Let $r_d(t), t \geq 0$, be a desired command signal and define the error state $\tilde{e}(t) \triangleq x(t) - r_d(t)$ so that the error dynamics are given by

$$\begin{aligned} m\ddot{\tilde{e}}(t) + c\dot{\tilde{e}}(t) + (k_1 + k_2(\tilde{e}^2(t) + 3r_d(t)\tilde{e}(t) + 3r_d^2(t)))\tilde{e}(t) &= bu(t) + \hat{d}w(t) \\ -(m\ddot{r}_d(t) + c\dot{r}_d(t) + k_1r_d(t) + k_2r_d^3(t)), \quad \tilde{e}(0) = \tilde{e}_0, \quad \dot{\tilde{e}}(0) = \dot{\tilde{e}}_0, \quad t \geq 0 \end{aligned} \quad (54)$$

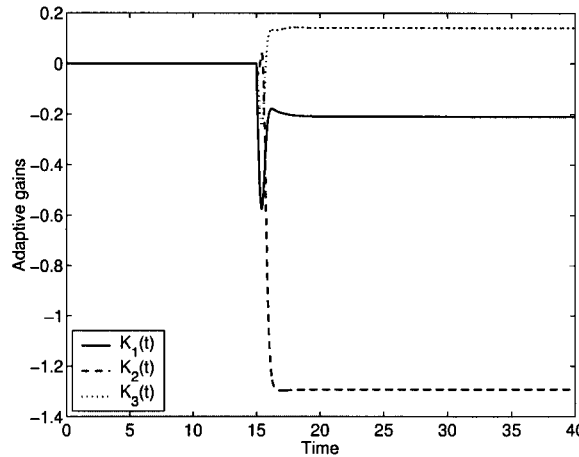


Figure 6. Adaptive gain history versus time.

Here, we assume that the disturbance signal $w(t)$ is a sinusoidal signal with unknown amplitude and phase; that is, $\hat{d}w(t) = \sqrt{A_1^2 + A_2^2} \sin(\omega t + \phi) = A_1 \sin \omega t + A_2 \cos \omega t$, where $\phi = \tan^{-1}(A_2/A_1)$ and A_1 and A_2 are unknown constants. Furthermore, the desired trajectory is given by

$$r_d(t) = \tanh\left(\frac{t - 20}{5}\right)$$

so that the position of the mass is moved from -1 to 1 at $t = 20$ s. Note that with $e_1 = \tilde{e}$ and $e_2 = \dot{\tilde{e}}$, (53) can be written in state-space form (15) with $e = [e_1, e_2]^T$, $f_i(r_d, e) = [e_2, -(1/m)(k_1 + k_2(e_1^2 + 3r_d e_1 + 3r_d^2))e_1 - (c/m)e_2]^T$, $G(t, e) = [0, (b/m)]^T$, $J_t(t, e) = 1/m[0_{6 \times 1}, \hat{d}_t^T]^T$, where $\hat{d}_t = [A_1, A_2, -k_1, -k_2, -c, -m]$, and $w_t(t) = [\sin \omega t, \cos \omega t, r_d(t), r_d^3(t), \dot{r}_d(t), \ddot{r}_d(t)]^T$. Here, we parameterize $f_i(r_d, e) = [e_2, \theta_1 e_1 + \theta_2 e_2 + \theta_3 e_1^3 + \theta_4 r_d e_1^2 + \theta_5 r_d^2 e_1]^T$, where $\theta_i, i = 1, \dots, 5$, are unknown constants. Next, let $G(t, e) = 1$, $F(r_d, e) = [e_1, e_2, e_1^3, r_d e_1^2, r_d^2 e_1]^T$, and $K_g = m/b[\theta_{n_1} - \theta_1, \theta_{n_2} - \theta_2, -\theta_3, -\theta_4, -\theta_5]$, where $\theta_{n_1}, \theta_{n_2}$ are arbitrary scalars, so that $f_c(e)$ is given by (50). Now, with the proper choice of θ_{n_1} and θ_{n_2} , it follows from Corollary 2.1 and Remark 2.3 that the adaptive feedback controller (26) guarantees that $e(t) \rightarrow 0$ as $t \rightarrow \infty$. Specifically, here we choose $\theta_{n_1} = -1, \theta_{n_2} = -2$, and $R = 2I_2$, so that P satisfying (21) is given by (51). With $m = 1, c = 1, k_1 = 2, k_2 = 0.5, \hat{d}w(t) = 2 \sin(\omega t + 1), \omega = 2, b = 3, Y = I_5, Z = I_6$, and initial conditions $e(0) = [0, 0]^T, K(0) = 0_{1 \times 5}$, and $\Phi(0) = 0_{1 \times 6}$, Figure 7 shows the actual position and the reference signal versus time and the control signal versus time. Finally, Figure 8 shows the adaptive gain history versus time.

Example 4.4

Consider the two-degree of freedom uncertain structural system given by

$$M_s \ddot{x}(t) + C_s \dot{x}(t) + K_s x(t) = u(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \quad t \geq 0 \tag{55}$$

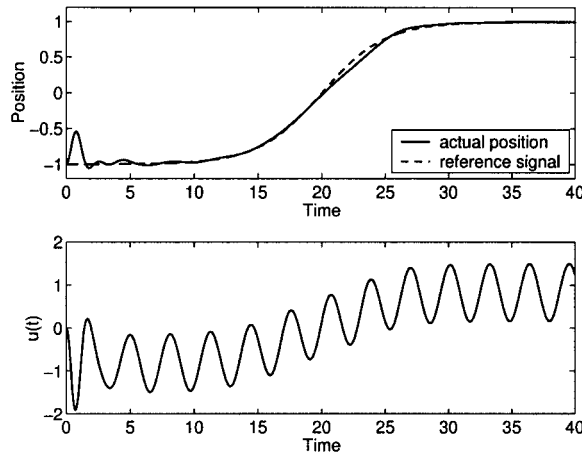


Figure 7. Position and control signal versus time.

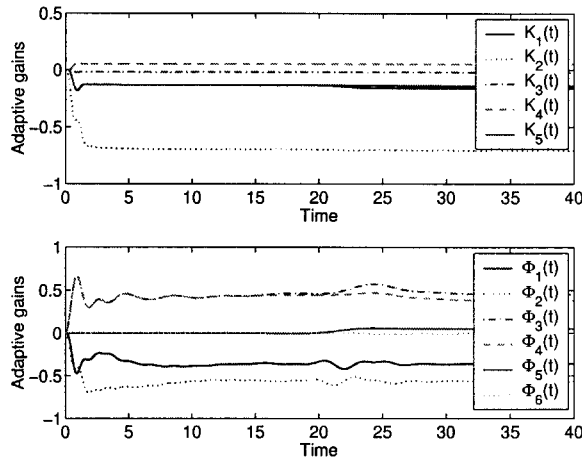


Figure 8. Adaptive gain history versus time.

where $x(t) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}^2$, $t \geq 0$,

$$M_s \triangleq \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad C_s \triangleq \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \quad K_s \triangleq \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

and $m_1, m_2, c_1, c_2, k_1, k_2 \in \mathbb{R}$ are positive unknown constants. Let $r_d(t)$ be a desired command signal and define the error state $\tilde{e}(t) \triangleq x(t) - r_d(t)$ so that the error dynamics are given by

$$M_s \ddot{\tilde{e}}(t) + C_s \dot{\tilde{e}}(t) + K_s \tilde{e}(t) = u(t) - M_s \ddot{r}_d(t) - C_s \dot{r}_d(t) - K_s r_d(t), \quad \tilde{e}(0) = \tilde{e}_0, \quad \dot{\tilde{e}}(0) = \dot{\tilde{e}}_0, \quad t \geq 0 \tag{56}$$

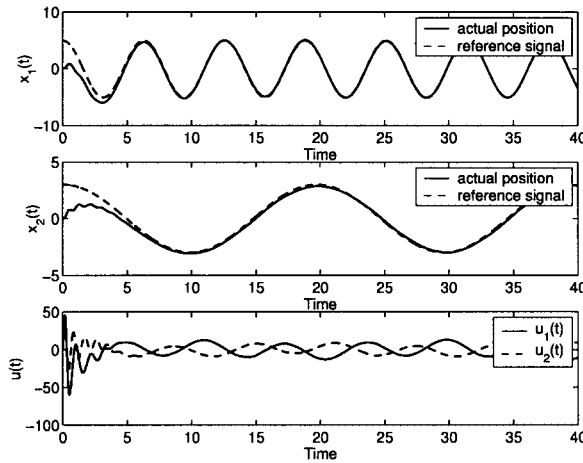


Figure 9. Positions and control signals versus time.

Note that with $e_1 = \tilde{e}$ and $e_2 = \dot{\tilde{e}}$, (59) can be written in state-space form (15) with $e = [e_1^T, e_2^T]^T$, $f_i(t, e) = [e_2^T, -(M_s^{-1}K_s e_1 + M_s^{-1}C_s e_2)^T]^T$, $G(t, e) = [0_{2 \times 2}, M_s^{-1}]^T$, $J_i(t, e) = [0_{6 \times 2}, \hat{D}_1^T]^T$, where $\hat{D}_1 = [-I_2, -M_s^{-1}C_s, -M_s^{-1}K_s]$ and $w_i(t) = [\dot{r}_d^T, \dot{r}_d^T, r_d^T]^T$. Note that M_s^{-1} is symmetric and positive definite but unknown. Here, we parameterize $f_i(t, e)$ as $f_i(t, e) = [e_2^T, (\Theta_1 e_1 + \Theta_2 e_2)^T]^T$, where $\Theta_1 \in \mathbb{R}^{2 \times 2}$ and $\Theta_2 \in \mathbb{R}^{2 \times 2}$ are unknown constant matrices. Next, let $\hat{G}(t, e) = I_2$, $F(t, e) = e$, and $K_g = M_s[\Theta_{n_1} + M_s^{-1}K_s, \Theta_{n_2} + M_s^{-1}C_s]$, where $\Theta_{n_1} \in \mathbb{R}^{2 \times 2}$, $\Theta_{n_2} \in \mathbb{R}^{2 \times 2}$ are arbitrary matrices, so that

$$f_c(e) = \begin{bmatrix} 0_2 & I_2 \\ \Theta_{n_1} & \Theta_{n_2} \end{bmatrix} e$$

Now, with the proper choice of Θ_{n_1} and Θ_{n_2} , it follows from Corollary 2.1 and Remark 2.3 that the adaptive feedback controller (26) guarantees that $e(t) \rightarrow 0$ as $t \rightarrow \infty$. Specifically, here we choose $\Theta_{n_1} = -I_2$, $\Theta_{n_2} = -I_2$, and $R = 2I_4$, so that P satisfying (21) is given by

$$P = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

With $m_1 = 3$, $m_2 = 2$, $c_1 = c_2 = 1$, $k_1 = 2$, $k_2 = 1$, $r_d(t) = [5 \cos(t), 3 \cos(t/\pi)]^T$, $Y = I_4$, $Z = I_6$, and initial conditions $e(0) = 0_{4 \times 1}$, $K(0) = 0_{2 \times 4}$, and $\Phi(0) = 0_{2 \times 6}$, Figure 9 shows the actual positions and the reference signals versus time and the control signals versus time. Finally, Figures 10 and 11 show the adaptive gain history versus time.

Example 4.5

The following example considers the utility of the proposed adaptive control framework for L_2 disturbance rejection. Specifically, consider the non-linear dynamical system representing

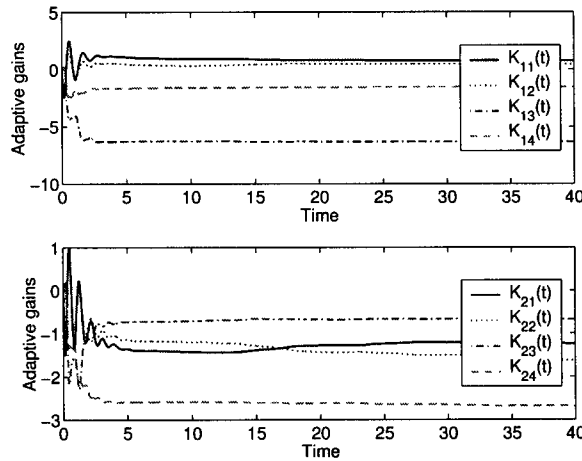


Figure 10. Adaptive gain history versus time.

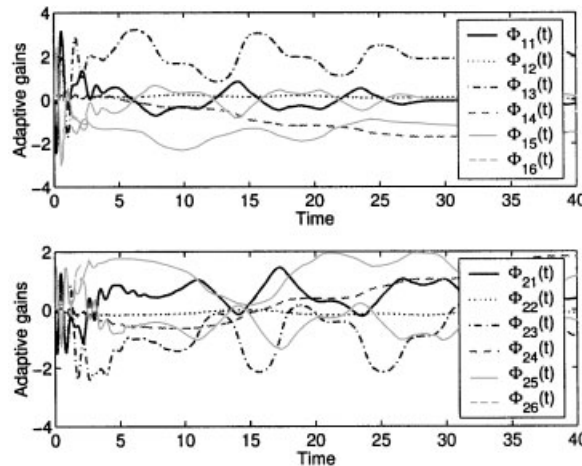


Figure 11. Adaptive gain history versus time.

a controlled rigid spacecraft given by

$$\dot{x}(t) = -I_b^{-1} X I_b x(t) + I_b^{-1} u(t) + D w(t), \quad x(0) = x_0, \quad w(\cdot) \in L_2, \quad t \geq 0 \quad (57)$$

where $x = [x_1, x_2, x_3]^T$ represents the angular velocities of the spacecraft with respect to the body-fixed frame, $I_b \in \mathbb{R}^{3 \times 3}$ is an unknown positive-definite inertia matrix of the spacecraft, $u = [u_1, u_2, u_3]^T$ is a control vector with control inputs providing body-fixed torques about three mutually perpendicular axes defining the body-fixed frame of the spacecraft, $D \in \mathbb{R}^{3 \times 1}$, and

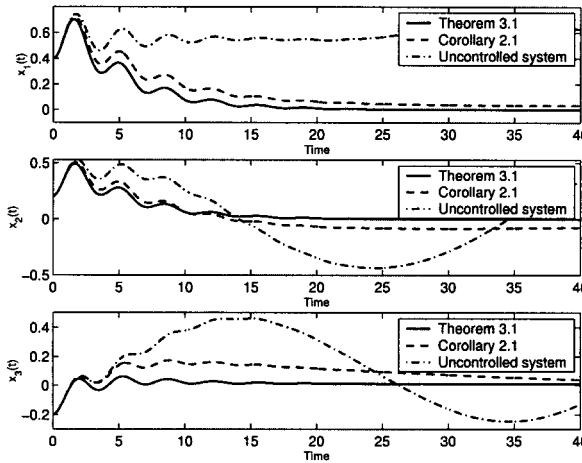


Figure 12. Angular velocities versus time.

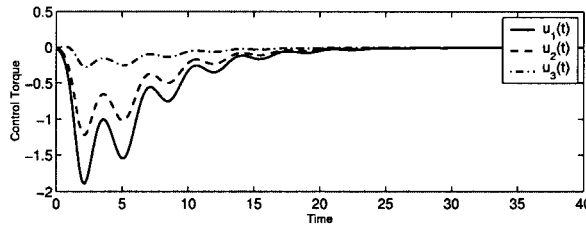


Figure 13. Control signals versus time.

X denotes the skew-symmetric matrix

$$X \triangleq \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Note that (57) can be written in state-space form (31) with $f(x) = -I_b^{-1} X I_b x$, $G(x) = I_b^{-1}$, and $J(x) = D$. Here, we assume that the inertia matrix I_b of the spacecraft is symmetric and positive definite but unknown. Since $f(x)$ is a quadratic function, we parameterize $f(x)$ as $f(x) = \Theta f_n(x)$, where $\Theta \in \mathbb{R}^{3 \times 6}$ is an unknown matrix and $f_n(x) = [x_1^2, x_2^2, x_3^2, x_1 x_2, x_2 x_3, x_3 x_1]^T$. Next, let $G_n(x) = I_3$, $F(x) = [f_n^T(x), x^T]^T$, and $K_g = I_b[-\Theta, \Phi_n]$, where $\Phi_n \in \mathbb{R}^{3 \times 3}$, is an arbitrary matrix, so that

$$f_c(x) = \Phi_n x = A_c x$$

Now, with the proper choice of Φ_n , it follows from Theorem 3.1 that the adaptive feedback controller (47) with update law (48) guarantees that $x(t) \rightarrow 0$ as $t \rightarrow \infty$ with $w(t) \equiv 0$. Further-

more, the closed-loop non-linear input–output map from L_2 disturbances $Dw(t)$ to performance variable $z(t) = Ex(t)$ satisfies the non-expansivity constraint (37). Here, we choose $A_c = -10I_3$, $E^T E = 2I_3$, and $\gamma = 1.4$, so that P satisfying (44) is given by

$$P = \begin{bmatrix} 0.1653 & 0.0408 & 0.0245 \\ 0.0408 & 0.1255 & 0.0153 \\ 0.0245 & 0.0153 & 0.1092 \end{bmatrix}$$

With

$$I_b = \begin{bmatrix} 20 & 0 & 0.9 \\ 0 & 17 & 0 \\ 0.9 & 0 & 15 \end{bmatrix}, \quad Y = 10I_9, \quad D = \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix}, \quad w(t) = e^{-0.2t} \sin 1.8t$$

and initial conditions $x(0) = [0.4, 0.2, -0.2]$, and $K(0) = 0_{3 \times 9}$, Figure 12 shows the angular velocities versus time. Figure 13 shows the control signals versus time. An alternative adaptive feedback controller that also does not require knowledge of the inertia of the space-craft is presented in Reference [15]. However, unlike the proposed controller, the adaptive controller presented in Reference [15] is tailored to the spacecraft attitude control problem.

5. CONCLUSION

A direct adaptive non-linear control framework for adaptive stabilization, disturbance rejection, and command following of multivariable non-linear uncertain systems with exogenous bounded disturbances was developed. Using Lyapunov methods the proposed framework was shown to guarantee partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Furthermore, in the case where the non-linear system is represented in normal form with input-to-state stable zero dynamics, the non-linear adaptive controllers were constructed without knowledge of the system dynamics. Finally, several illustrative numerical examples were presented to show the utility of the proposed adaptive stabilization and tracking scheme.

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