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Journal of the Franklin Institute 340 (2003) 371–390

Journal of The Franklin Institute

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# Linear controller analysis and design for systems with input hystereses nonlinearities ☆

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Received 1 October 2000; received in revised form 1 August 2003; accepted 26 August 2003

**Abstract**

An output feedback control analysis and design framework for linear systems with input hystereses nonlinearities is developed. Specifically, by transforming the hystereses nonlinearities into dissipative input–output dynamical operators, dissipativity theory is used to analyze and design linear controllers for systems with hysteretic actuators. The overall framework guarantees partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system state associated with the plant and the controller. Furthermore, the remainder of the state associated with the hysteresis dynamics is shown to be semistable; that is, solutions of the hysteretic system converge to Lyapunov stable equilibrium points determined by the system initial conditions.


*Keywords:* Hystereses nonlinearities; Preisach model; Linear stabilization; Output feedback; Dissipativity theory; Semistability; Partial stability

## 1. Introduction

In recent years the desire to orbit large, lightweight space structures with high-performance requirements has prompted researchers to consider actuators which

☆ This research was supported in part by the National Science Foundation under Grant ECS-9496249 and the Air Force Office of Scientific Research under Grant F49620-001-0095.

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0016-0032/$30.00 
doi:10.1016/j.jfranklin.2003.08.002

possess a fraction of the size and weight of more conventional actuation devices. As a consequence, considerable research interest has focused in the field of smart or adaptive materials as a viable alternative to conventional proof mass actuators for vibration control [1–6]. Due to the fact that adaptation in smart materials is a result of physical nonlinear changes occurring within the material, these actuation devices exhibit significant hysteresis in the actuator response. Specifically, smart distributed actuators such as shape memory alloys, magnetostrictives, electrorheological fluids, and piezoceramics all exhibit hysteretic effects [1–6]. Since hystereses nonlinearities can severely degrade closed-loop system performance, and in some cases drive the system to a limit cycle instability, they must be accounted for in the control-system design process.

Even though numerous models for capturing hystereses effects have been developed [7–14], with the Preisach model [7–9] being the most widely used, controller analysis and synthesis for feedback systems with hystereses nonlinearities has received little attention in the literature. Notable exceptions include [2,11,13,15,16]. The main complexity arising in hystereses nonlinearities is the fact that every reachable point in the input–output hysteresis map does not correspond to a uniquely defined point. In fact, at any reachable point in the input–output hysteresis map there exists an infinite number of trajectories that may represent the future behavior of the hysteresis dynamics. These trajectories depend on a particular past history of the extremum values of the input. However, hystereses nonlinearities with counterclockwise loops have been shown to be dissipative with respect to a supply rate involving force inputs and velocity outputs [17]. Dissipative hystereses models include the well-known backlash nonlinearities, stiction nonlinearities, relay hystereses, and most of the hystereses nonlinearities arising in smart material actuators [15].

The contribution of this paper is a methodology for analyzing and designing output feedback controllers for systems with input hystereses nonlinearities. Specifically, by transforming the hystereses nonlinearities into dissipative input–output dynamic operators, dissipativity theory [18] is used to analyze and design linear controllers for systems with input hystereses nonlinearities. In particular, by representing the input hysteresis nonlinearity as a dissipative input–output dynamical operator with respect to a given supply rate, partial closed-loop asymptotic stability [19,20]; that is, asymptotic stability with respect to part of the closed-loop state associated with the plant and the controller, is guaranteed in the face of an input hysteresis nonlinearity. Furthermore, it is shown that the reminder of the state associated with the hysteresis dynamics is semistable [21]; that is, the limit points of the hysteretic states converge to Lyapunov stable equilibrium points determined by the system initial conditions.

## 2. Mathematical preliminaries

In this section, we establish definitions, notation, and several key results. Let $\mathscr{F}$ and $\hat{\mathscr{F}}$ denote real separable function spaces and let $\mathscr{B}(\mathscr{F}, \hat{\mathscr{F}})$ denote the space of bounded linear operators from $\mathscr{F}$ into $\hat{\mathscr{F}}$. In the case where $\mathscr{F} = \mathbb{R}^n$ and $\hat{\mathscr{F}} = \mathbb{R}^m$ are finite dimensional Euclidean spaces, we use the notation and terminology introduced above with appropriate changes. Specifically, bounded linear operators are represented by matrices over a fixed orthonormal basis so that $\mathbb{R}^{m\times n} = \mathscr{B}(\mathbb{R}^n, \mathbb{R}^m)$. Furthermore, for $M \in \mathbb{R}^{m\times n}$, we write $M^{\mathrm{T}}$ for the transpose of $M$ and $M \geqslant 0$ (resp., $M > 0$) to denote the fact that the square symmetric matrix $M$ is nonnegative (resp., positive) definite. Finally, we write $I_n$ to denote the $n \times n$ identity matrix, $V'(x)$ to denote the Fréchet derivative of $V$ at $x$, $\mathrm{C}^0$ to denote the set of continuous functions, and $\mathrm{C}^r$ to denote the set of functions with $r$ continuous derivatives.

In this paper, we represent nonlinear dynamical systems $\mathscr{G}$ defined on the semi-infinite interval $[0, \infty)$ as a mapping between function spaces satisfying an appropriate set of axioms. For the following definition, $\mathscr{U}$ is an input space and is a subset of bounded continuous $U$-valued functions on $[0, \infty)$. The set $U \subseteq \mathbb{R}^m$ contains the set of input values; that is, at any time $t$, $u(t) \in U$. The space $\mathscr{U}$ is assumed to be closed under the shift operator; that is, if $u \in \mathscr{U}$, then the function $u_T$ defined by $u_T(t) = u(t+T)$ is contained in $\mathscr{U}$ for all $T \geqslant 0$. Furthermore, $\mathscr{Y}$ is an output space and is a subset of continuous $Y$-valued functions on $[0, \infty)$. The set $Y \subseteq \mathbb{R}^l$ contains the set of output values; that is, each value of $y(t) \in Y$, $t \geqslant 0$. The space $\mathscr{Y}$ is assumed to be closed under the shift operator; that is, if $y \in \mathscr{Y}$, then the function $y_T$ defined by $y_T(t) = y(t+T)$ is contained in $\mathscr{Y}$ for all $T \geqslant 0$. Finally, $\mathscr{D}$ is a metric space with topology of uniform convergence and metric $\rho : \mathscr{D} \times \mathscr{D} \to [0, \infty)$. Hence, the notions of openness, convergence, continuity, and compactness that we use in the paper refer to the topology generated on $\mathscr{D}$ by the metric $\rho(\cdot, \cdot)$.

**Definition 2.1** (Willems [18]). A *stationary dynamical system* on $\mathscr{D}$ is the octuple $(\mathscr{D}, \mathscr{U}, U, \mathscr{Y}, Y, [0, \infty), s, q)$, where $s : [0, \infty) \times \mathscr{D} \times \mathscr{U} \to \mathscr{D}$ and $q : \mathscr{D} \times U \to Y$ are such that the following axioms hold:

(i) *Continuity*: $s(\cdot, \cdot, u)$ is jointly continuous for all $u \in \mathscr{U}$.
(ii) *Consistency*: $s(0, x_0, u) = x_0$ for all $x_0 \in \mathscr{D}$ and $u \in \mathscr{U}$.
(iii) *Determinism*: $s(t, x_0, u_1) = s(t, x_0, u_2)$ for all $t \in [0, \infty)$, $x_0 \in \mathscr{D}$, and $u_1, u_2 \in \mathscr{U}$ satisfying $u_1(\tau) = u_2(\tau)$, $\tau \leqslant t$.
(iv) *Semi-group property*: $s(\tau, s(t, x_0, u), u) = s(t+\tau, x_0, u)$ for all $x_0 \in \mathscr{D}$, $u \in \mathscr{U}$, and $\tau, t \in [0, \infty)$.
(v) *Read-out map*: There exists $y \in \mathscr{Y}$ such that $y(t) = q(s(t, x_0, u), u(t))$ for all $x_0 \in \mathscr{D}$, $u \in \mathscr{U}$, and $t \geqslant 0$.

Henceforth, we denote the dynamical system $(\mathscr{D}, \mathscr{U}, U, \mathscr{Y}, Y, [0, \infty), s, q)$ by $\mathscr{G}$. Furthermore, we refer to $s(t, x_0, u)$, $t \geqslant 0$, as the *trajectory* or *state transition operator* of $\mathscr{G}$ corresponding to $x_0 \in \mathscr{D}$ and $u \in \mathscr{U}$. For a given trajectory $s(t, x_0, u)$, $t \geqslant 0$, we refer to $x_0 \in \mathscr{D}$ as the *initial condition* of $\mathscr{G}$. Furthermore, an *equilibrium point* of the undisturbed ($u \equiv 0$) dynamical system $\mathscr{G}$ is a point $x \in \mathscr{D}$ satisfying $s(t, x, 0) = x$ for all $t \geqslant 0$. An equilibrium point $x \in \mathscr{D}$ of the undisturbed ($u \equiv 0$) dynamical system $\mathscr{G}$ is

*Lyapunov stable* if, for each open neighborhood $\mathcal{O}_1$ containing $x$, there exists an open neighborhood $\mathcal{O}_2 \subseteq \mathcal{O}_1$ containing $x$ such that $s(t, x_0, 0) \in \mathcal{O}_1$, $t \geqslant 0$, for all $x_0 \in \mathcal{O}_2$. An equilibrium point in $x \in \mathcal{D}$ of the undisturbed ($u \equiv 0$) dynamical system $\mathcal{G}$ is *semistable* [21] if $x$ is Lyapunov stable and there exists an open neighborhood $\mathcal{O}_3$ containing $x$ such that $\lim_{t \to \infty} s(t, x_0, 0)$ exists for all $x_0 \in \mathcal{O}_3$. For the dynamical system $\mathcal{G}$ given by Definition 2.1, a function $r : U \times Y \to \mathbb{R}$ is called a *supply rate* [18] if it is locally integrable; that is, for all input–output pairs $u \in U$ and $y \in Y$, $r(\cdot, \cdot)$ satisfies $\int_{t_1}^{t_2} |r(u(s), y(s))| \mathrm{d}s < \infty$, $t_1, t_2 \geqslant 0$.

**Definition 2.2** (Willems [18]). A dynamical system $\mathcal{G}$ is *dissipative with respect to the supply rate* $r(u, y)$ if there exists a continuous nonnegative-definite function $V_{\mathrm{s}} : \mathcal{D} \to \mathbb{R}$, called a *storage function*, such that the *dissipation inequality*

$$V_{\mathrm{s}}(x(t)) \leqslant V_{\mathrm{s}}(x(t_1)) + \int_{t_1}^{t} r(u(s), y(s))\, \mathrm{d}s \tag{1}$$

is satisfied for all $t_1, t \geqslant 0$ and where $x(t) = s(t, x_0, u(t))$, $t \geqslant t_1$, with $x_0 \in \mathcal{D}$ and $u(t) \in U$.

**Definition 2.3.** A set $\mathcal{M} \subset \mathcal{D}$ is a *positively invariant set* for the undisturbed ($u(t) \equiv 0$) dynamical system $\mathcal{G}$ if $x_0 \in \mathcal{M}$ implies that $s(t, x_0, 0) \in \mathcal{M}$ for all $t \geqslant 0$.

For the next result, define the notation

$$V^{-1}(\gamma) \triangleq \{x \in \mathcal{Q} : V(x) = \gamma\},$$

where $\gamma \in \mathbb{R}$, $\mathcal{Q} \subseteq \mathcal{D}$, and $V : \mathcal{Q} \to \mathbb{R}$ is a continuous function, and let $\mathcal{M}_\gamma$ denote the largest positively invariant set (with respect to the undisturbed ($u(t) \equiv 0$) dynamical system $\mathcal{G}$) contained in $V^{-1}(\gamma)$.

**Theorem 2.1** (LaSalle [22]). *Let* $s(t, x_0, 0)$, $t \geqslant 0$, *denote a trajectory of the nonlinear undisturbed* ($u(t) \equiv 0$) *dynamical system* $\mathcal{G}$ *and let* $\mathcal{D}_{\mathrm{c}} \subset \mathcal{D}$ *be a positively invariant set with respect to* $\mathcal{G}$. *Assume that every positive orbit* $\gamma^+(x_0) \triangleq \bigcup_{t \in [0, \infty)} s(t, x_0, 0)$ *in* $\mathcal{D}_{\mathrm{c}}$ *is contained in a compact set and there exists a continuous function* $V : \mathcal{D}_{\mathrm{c}} \to \mathbb{R}$ *such that* $V(s(t, x_0, 0)) \leqslant V(s(\tau, x_0, 0))$, $0 \leqslant \tau \leqslant t$, *for all* $x_0 \in \mathcal{D}_{\mathrm{c}}$. *If* $x_0 \in \mathcal{D}_{\mathrm{c}}$, *then* $s(t, x_0, 0) \to \mathcal{M} \triangleq \bigcup_{\gamma \in \mathbb{R}} \mathcal{M}_\gamma$ *as* $t \to \infty$.

Next, we consider finite dimensional linear dynamical systems. Specifically, consider the linear system with a state space representation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geqslant 0, \tag{2}$$

$$y(t) = Cx(t) + Du(t), \tag{3}$$

and transfer function $G(s) = C(sI_n - A)^{-1}B + D$, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, and $D \in \mathbb{R}^{m \times m}$. Let

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

denote a state space realization of $G(s)$. The notation $\overset{\min}{\sim}$ is used to denote a minimal realization. A square transfer function $G(s)$ is called *positive real* [23] if (1) all the poles of $G(s)$ lie in the closed left-half plane with simple poles on $s = j\omega$, and (2) $G(s) + G^*(s)$ is nonnegative definite for $\mathrm{Re}[s] > 0$. A square transfer function $G(s)$ is called *strictly positive real* [23] if there exists $\varepsilon > 0$ such that $G(s - \varepsilon)$ is positive real.

Next, we state the strict positive real lemma used to characterize strict positive realness in the state space setting.

**Lemma 2.1** (Khalil [23]).

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

*is strictly positive real if and only if there exist matrices* $P \in \mathbb{R}^{n \times n}$, $L \in \mathbb{R}^{p \times n}$, *and* $W \in \mathbb{R}^{p \times m}$, *with* $P$ *positive definite, and a positive constant* $\varepsilon$ *such that*

$$0 = A^{\mathrm{T}}P + PA + \varepsilon P + L^{\mathrm{T}}L, \tag{4}$$

$$0 = PB - C^{\mathrm{T}} + L^{\mathrm{T}}W, \tag{5}$$

$$0 = D + D^{\mathrm{T}} - W^{\mathrm{T}}W. \tag{6}$$

Finally, the following theorem is needed for the main results in this paper.

**Theorem 2.2.** *Let*

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

*be asymptotically stable. Assume that every element of* $G(s)$ *has at least one zero at the origin; that is,* $G(0) = -CA^{-1}B + D = 0$. *Then* $s^{-1}G(s)$ *is strictly positive real if and only if there exist matrices* $P \in \mathbb{R}^{n \times n}$ *and* $L \in \mathbb{R}^{p \times n}$, *with* $P$ *positive definite, and a positive constant* $\varepsilon$ *such that*

$$0 = A^{\mathrm{T}}P + PA + \varepsilon P + L^{\mathrm{T}}L, \tag{7}$$

$$0 = B^{\mathrm{T}}P - CA^{-1}. \tag{8}$$

**Proof.** The proof is a direct consequence of Lemma 2.1 using $-CA^{-1}B + D = 0$ and noting that $s^{-1}G(s) = CA^{-1}(sI - A)^{-1}B$. □

## 3. Preisach dynamical model for hystereses nonlinearities

In this section, we introduce a mathematical model due to Preisach [7] for capturing the dynamics of hystereses nonlinearities. Specifically, let $\sigma_{\mathrm{h}}(\cdot)$ denote a scalar rate-independent, counterclockwise hysteresis nonlinearity with dynamic memory. Rate-independent refers to the fact that the hysteresis curves generated in $\mathbb{R}^2$ by an input–output pair are input rate-independent, while counterclockwise refers to the fact that the loops generated by hystereses nonlinearities are counterclockwise. For a given bounded continuous input $u(t) \in \mathbb{R}$, $t \geqslant 0$, the output of a Preisach hysteresis model is given by

$$y(t) = \sigma_{\mathrm{h}}(u(t)) = \int\int_{\alpha \geqslant \beta} \mu(\alpha, \beta)\, \gamma_{\alpha\beta}[u(t)]\, \mathrm{d}\alpha\, \mathrm{d}\beta, \tag{9}$$

where $\gamma_{\alpha\beta}(\cdot)$ is a unit relay characterized by two switching points $\alpha$ and $\beta$ as shown in Fig. 1, and $\mu(\alpha, \beta) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a weighting function associated with each unit relay.

If $\mu(\cdot, \cdot)$ is bounded and piecewise continuous, then $\sigma_{\mathrm{h}} : C^0 \to C^0$. If, in addition, $\mu(\cdot, \cdot)$ is nonnegative definite and has a finite support, then $\sigma_{\mathrm{h}} : \mathscr{S} \to \mathscr{S}$, where $\mathscr{S}$ denotes the real separable Sobolev space $\mathscr{S} \triangleq \{u(\cdot) \in C^1 : \int_{-\infty}^{\infty} (u^2(t) + \dot{u}^2(t))\mathrm{d}t < \infty\}$. For details of these facts see Ref. [24]. The Preisach model characterized by Eq. (9) reflects the fact that a hysteresis nonlinearity can be modeled as a superposition of independent weighted relays with different switching points. Note that to guarantee counterclockwise hysteretic loops we require $\alpha \geqslant \beta$. Although $\gamma_{\alpha\beta}[u(t)]$ in Eq. (9) has local memory; that is, it only depends on certain recent portions of the input time history, Preisach models at a given time $t$ are generally dependent on the overall input time history and hence are dynamic. This fact is elucidated below.

Most mathematical properties of the nonlinear dynamical Preisach model are conveniently facilitated by geometrical interpretation. Specifically, consider the half $(\alpha, \beta)$-plane $\mathscr{P}$ (Preisach plane) shown in Fig. 2(a) whose points represent unit relays with associated $(\alpha, \beta)$ switching point pairs. Since every point in $\mathscr{P}$ can only have two

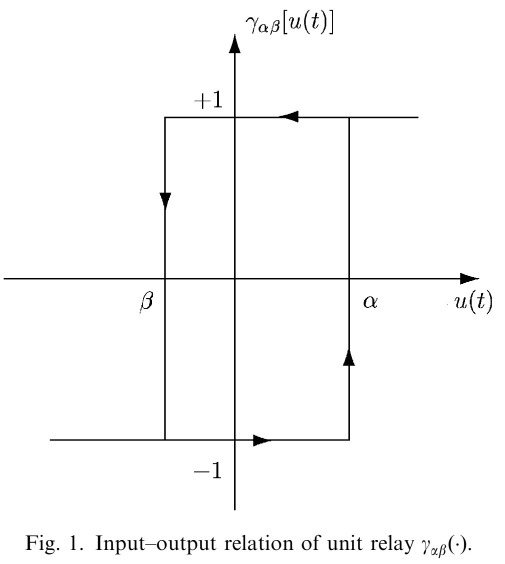


Fig. 1. Input–output relation of unit relay $\gamma_{\alpha\beta}(\cdot)$.

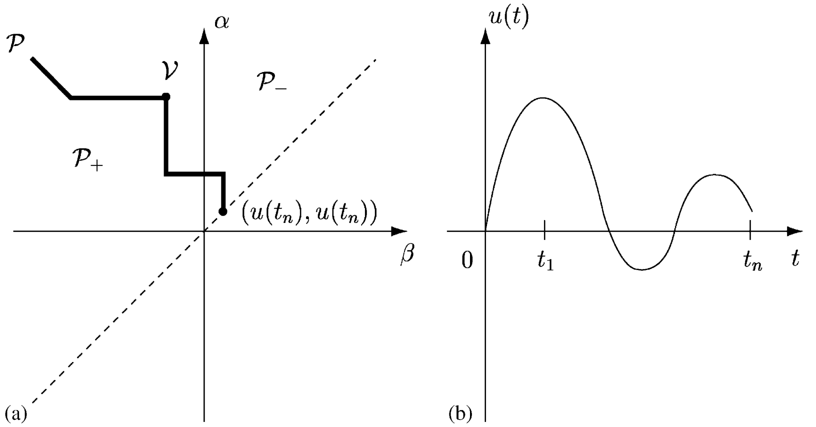


Fig. 2. Moving boundary in the Preisach plane $\mathscr{P}$ with input $u(t)$, $t \geqslant 0$.

output values; namely $+1$ or $-1$, $\mathscr{P}$ is divided into two input-dependent regions $\mathscr{P}_+(t)$ and $\mathscr{P}_-(t)$ in which $\gamma_{\alpha\beta}[u(t)]$ is $+1$ and $-1$, respectively. Note that $\mathscr{P}_+(t) \cup \mathscr{P}_-(t) = \mathscr{P}$ for all $t$, and hence Eq. (9) can be written as

$$y(t) = \sigma_{\mathrm{h}}(u(t)) = \int\int_{\mathscr{P}_+(t)} \mu(\alpha, \beta)\, \mathrm{d}\alpha\, \mathrm{d}\beta - \int\int_{\mathscr{P}_-(t)} \mu(\alpha, \beta)\, \mathrm{d}\alpha\, \mathrm{d}\beta. \tag{10}$$

These input-dependent regions result in the dynamic nature of the Preisach model. To see this, assume that the system starts at a *demagnetized* state [8] at time $t = -T$; that is, $\mathscr{P}_+(t)$ and $\mathscr{P}_-(t)$ are initially divided along the line $\alpha = -\beta$ at time $t = -T$, $T > 0$, and the initial configuration of $\mathscr{P}_+(t)$ and $\mathscr{P}_-(t)$ at $t = 0$ shown in Fig. 2(a) is attained by a continuous (fictitious) input defined on $(-T, 0]$. Next, suppose a bounded continuous input $u(t)$, $t \geqslant 0$, is applied to $\sigma_{\mathrm{h}}(\cdot)$ as shown in Fig. 2(b). Note that since $u(t)$, $t \geqslant 0$, is continuous and bounded, $\mathscr{P}_+(t)$ and $\mathscr{P}_-(t)$ are simply connected regions. Now, if $u(t)$, $t \geqslant 0$, is increasing, relays in $\mathscr{P}_-(t)$ with $\alpha \leqslant u(t)$, $t \geqslant 0$, will switch to $+1$ and $\mathscr{P}_+(t)$ will increase. Furthermore, the horizontal branch of the boundary (Preisach boundary) shown in Fig. 2(a) will move along the $\alpha$-axis in a positive direction. Alternatively, if $u(t)$, $t \geqslant t_1$, is decreasing, the input will switch relays with $\beta \geqslant u(t)$, $t \geqslant t_1$, to $-1$ and $\mathscr{P}_-(t)$ will increase. In this case, the vertical branch of the Preisach boundary shown in Fig. 2(a) will move along the $\beta$-axis in a negative direction and define a vertex $\mathscr{V}$ on the boundary. For every input reversal applied to the Preisach model, a corresponding vertex on the boundary is defined which gives the Preisach boundary a decreasing staircase shape as shown in Fig. 2(a). However, if the input magnitude at a given time is greater than past input extrema, some of the vertices can disappear. This phenomenon is known as the *wiping out* property [8] and is prevalent in Preisach hystereses models. In addition, the final branch of the moving Preisach boundary always touches the line $\alpha = \beta$ so that the intersection between the boundary and the line $\alpha = \beta$ in the Preisach plane $\mathscr{P}$ is $(u(t), u(t))$ for all $t$. Finally, it is important to note that the remaining vertices and the current value of the input on the moving boundary uniquely define the output of the model.

The above geometrical interpretation of the Preisach model shows that the output $y(t) = \sigma_h(u(t))$ of the hysteresis nonlinearity is a function of the input $u(t)$, $t \geqslant 0$, past extrema of the input, and the initial state of the Preisach boundary at $t = 0$. Furthermore, the input-dependent Preisach boundary plays a key role in the memory of these past input extrema. The above observations show that the Preisach model is a dynamical model that captures hystereses nonlinearities.

To investigate the dynamical properties of the Preisach model, let $\mathcal{U}$ be the set of bounded inputs defined by

$$\mathcal{U} \triangleq \{u \in C^0 : u(t) \in U \subseteq \mathbb{R}, t \geqslant -T : |u(t)| < M, t \geqslant -T, \quad \text{and} \quad \lim_{t \to -T} u(t) = 0\},$$

where $M > 0$, let the output space $\mathcal{Y}$ be the set of real-valued continuous functions defined on $[0, \infty)$, and let $Y$ be the set of outputs given by $y(t) = \sigma_h(u(t))$, $u(t) \in U$, $t \geqslant 0$. Furthermore, define the metric space $\mathcal{D}$ as a set of all Lipschitz continuous functions $\psi : [0, M] \to \mathbb{R}$ denoting the Preisach boundary with the metric $\rho(x_1, x_2) \triangleq \int_0^M |x_1(s) - x_2(s)| \mathrm{d}s$. Note that $\rho(x_1, x_2) = \int\int_{\mathcal{P}_{x_1x_2}} \mathrm{d}\alpha\, \mathrm{d}\beta$, where $\mathcal{P}_{x_1x_2} \subset \mathcal{P}$ is the region bounded by $x_1, x_2 \in \mathcal{D}$ and the $\alpha = \beta$ line as shown in Fig. 3. Finally, we assign $\mathcal{U}$, $\mathcal{Y}$, and $\mathcal{D}$ as the input space, the output space, and the state space, respectively, and define the state transition operator $s(\cdot, \cdot, \cdot)$ of the Preisach model as a concatenation of mappings between these spaces. In particular, the moving Preisach boundary in $\mathcal{D}$ is taken as the state of the (infinite dimensional) system. Since the moving Preisach boundary in the Preisach model has a decreasing staircase shape, there exists a one-to-one mapping from $\mathcal{D}$ into a set of sequence of past input extrema or, equivalently, the vertices of the moving Preisach boundary. This sequence is called the *reduced memory sequence* and is defined as follows.

**Definition 3.1** (Visintin [25]). Let $u \in \mathcal{U}$. Then the *reduced memory sequence* of $u(t) \in U$, $t \geqslant -T$, denoted by $\{r_i\}_{i=0}^{\infty}$ is an infinite sequence of non-wiped out extrema

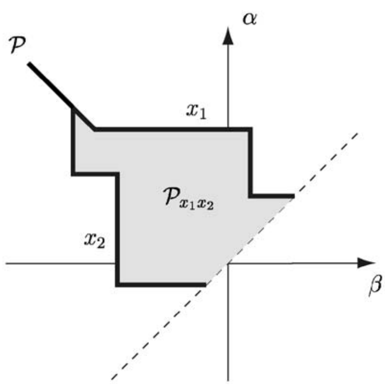


Fig. 3. Bounded region $\mathcal{P}_{x_1x_2} \subset \mathcal{P}$.

constructed from the past history of $u(t) \in U$, $t \geqslant -T$, satisfying

$$r_0 = 0,$$
$$r_1 = u(t_1),$$
$$r_i = \begin{cases} \max_{\tau \in (t_i, t]} u(\tau) & \text{if } r_{i-1} < r_{i-2}, \\ \min_{\tau \in (t_i, t]} u(\tau) & \text{if } r_{i-1} > r_{i-2}, \end{cases}$$

where $t_1 = \max\{\tau \in (-T, t] : |u(t)| = \eta\}$, $\eta = \max_{\tau \in (-T, t]} |u(t)|$, and $t_i = \max\{\tau \in (t_{i-1}, t] : u(\tau) = r_i\}$. If $u \in \mathscr{U}$ has a finite number of extrema, the reduced memory sequence has finite length $N$, $t_N = t$, and $u(t_N) = u(\tau)$. In this case, $u(t)$, $t \geqslant -T$, has a finite number of non-wiped out past extrema, and hence set $r_i = r_N$ for all $i \geqslant N$.

Let $\mathscr{R} \subset \ell_\infty$, where $\ell_\infty$ is the set of all bounded sequences, be the space of reduced memory sequences of $u \in \mathscr{U}$. Note that the one-to-one mapping between $\mathscr{R}$ and $\mathscr{D}$ implies that the Preisach model is an infinite dimensional dynamical system. Now, let $\phi_1 : \mathscr{U} \to \mathscr{R}$, $\phi_2 : \mathscr{R} \to \mathscr{D}$, and $q : \mathscr{D} \to \mathscr{Y}$ denote the mappings between the input space $\mathscr{U}$, the reduced memory sequence space $\mathscr{R}$, the state space $\mathscr{D}$, and the output space; that is,

$$\mathscr{U} \underset{\phi_1^{\mathrm{r}}}{\overset{\phi_1}{\rightleftarrows}} \mathscr{R} \underset{\phi_2^{-1}}{\overset{\phi_2}{\rightleftarrows}} \mathscr{D} \overset{q}{\to} \mathscr{Y}, \tag{11}$$

where $\phi_1 : \mathscr{U} \to \mathscr{R}$ is given by the algorithm in Definition 3.1. Note that since the reduced memory sequence only identifies dominant extrema of $u \in \mathscr{U}$, it follows that an infinite number of inputs have the same reduced memory sequence. Hence, $\phi_1^{-1}$ does not exist. Thus, the mapping from the reduced memory sequence space $\mathscr{R}$ to the input space is given by a right inverse $\phi_1^{\mathrm{r}} : \mathscr{R} \to \mathscr{U}$. For further details of these mappings see Ref. [24]. Now, for a given initial condition $x_{\mathrm{h}}(0) \in \mathscr{D}$ and input $u \in \mathscr{U}$, the state transition operator $s(\cdot,\cdot,\cdot)$ for the Preisach model is given by

$$s(t, x_{\mathrm{h}}(0), u(t)) = \phi_2 \circ \phi_1(\phi_1^{\mathrm{r}} \circ \phi_2^{-1}(x_{\mathrm{h}}(0)) \diamond u(t)), \tag{12}$$

where $\circ$ denotes the composition operator and $\diamond : \mathscr{S} \times \mathscr{S} \to \mathscr{S}$ denotes the concatenation operator defined by

$$u_{(t_0, t_1]} \diamond v_{(t_1, t_2]} \triangleq \begin{cases} u, & t_0 < t \leqslant t_1, \\ v, & t_1 < t \leqslant t_2. \end{cases} \tag{13}$$

Note that $s(t, x_{\mathrm{h}}(0), u(t))$ first maps the given initial condition $x_{\mathrm{h}}(0)$ to a fictitious input function in $\mathscr{U}$ defined on $(-T, 0]$ to generate an initial configuration (initial condition) of the Preisach boundary that does *not* correspond to a demagnetized state. This map is then concatenated with a given input $u(t)$ defined on $(0, t]$. The resulting map is then mapped to $\mathscr{D}$. As shown in Ref. [24], the state transition operator given by Eq. (12) satisfies the axioms in Definition 2.1 and hence the Preisach model is a dynamical system.

The following result allows us to characterize the set of all equilibria for a Preisach model. Let $\mathscr{D}_{\mathrm{e}} \triangleq \{x \in \mathscr{D} :$ there exists $t > 0$ and $u : [-T, t) \to \mathscr{U}$ such that $u(t) = 0$, $t \geqslant 0$,

$x = s(t, x_h(0), u(t))$, and $x_h(0) = \phi_2(\phi_1(u(0)))\}$; that is, $\mathscr{D}_e$ is the set of all trajectories in $\mathscr{D}$ such that $u(t) = 0$, $t \geqslant 0$, and there exists a (fictitious) input $u : [-T, t) \to \mathbb{R}$ such that the state transition operator starting at a demagnetized state at $t = -T$ satisfies $s(t, x_h(0), u(t)) \in \mathscr{D}_e$ for some time $t > 0$.

**Proposition 3.1.** *Let $x_h(0) \in \mathscr{D}_e$. Then $s(t, x_h(0), 0) = x_h(0)$, for all $t \geqslant 0$.*

**Proof.** Let $u_0 = \phi_1^r \circ \phi_2^{-1}(x_h(0))$ and let $0_{(t_1,t_2]}$ denote the zero function on $(t_1, t_2]$. Note that $u_0$ is defined on $(-T, 0]$ and $u_0(0) = 0$ since $x_h(0) \in \mathscr{D}_e$. Now, for any $t \geqslant 0$, $\max_{\tau \in (t_n,t]}(u_0 \diamond 0_{(0,t]}) = \max_{\tau \in (t_n,t]} u_0(\tau)$ and $\min_{\tau \in (t_n,t]}(u_0 \diamond 0_{(0,t]}) = \min_{\tau \in (t_n,t]} u_0(\tau)$, for all $t_n \leqslant t$. Hence, it follows from the definition of the reduced memory sequence that $\phi_1(u_0 \diamond 0_{(0,t]}) = \phi_1(u_0)$. Now, the result follows by noting that $s(t, x_h(0), 0) = \phi_2 \circ \phi_1(u_0 \diamond 0_{(0,t]}) = \phi_2 \circ \phi_1(u_0) = \phi_2 \circ \phi_1 \circ \phi_1^r \circ \phi_2^{-1}(x_h(0)) = x_h(0)$, $t \geqslant 0$. $\square$

Proposition 3.1 shows that $\mathscr{D}_e$ corresponds to the set of all equilibria for a Preisach model. Furthermore, note that there exists an infinite number of states in $\mathscr{D}_e$ since every decreasing Preisach boundary that touches the origin of the Preisach plane $\mathscr{P}$ is an element of $\mathscr{D}_e$. Finally, since that the set of all trajectories $s(t, x_h(0), u(t)) \in \mathscr{D}$ with $u(t) = 0$ for all $t \geqslant 0$ also belong to $\mathscr{D}_e$, it follows from Proposition 3.1 that in the absence of external input; the state space of a Preisach dynamical system consists of only equilibria and hence is semistable.

The following properties of the metric space $\mathscr{D}$ (with metric $\rho$) and the state transition operator of the Preisach model are needed for the main stability results of this paper.

**Lemma 3.1** (Gorbet [24]). *The metric space $\mathscr{D}$ with metric $\rho : \mathscr{D} \times \mathscr{D} \to [0, \infty)$ is complete.*

**Proposition 3.2.** *Let $u \in \mathscr{U}$ and suppose there exists $\varepsilon > 0$ and $t_N \geqslant 0$ such that $|u(t)| < \varepsilon$ for all $u(t) \in U$, $t > t_N$. Then $\rho(s(t_p, x_h(0), u(t_p)), s(t_q, x_h(0), u(t_q))) < 2\varepsilon M$, $t_p, t_q > t_N$, $x_h(0) \in \mathscr{D}$.*

**Proof.** Let $t_p > t_q$. To determine the metric between two hysteretic states in a given orbit, consider all points in the Preisach plane $\mathscr{P}$ such that $\gamma_{\alpha\beta}(t_p) \neq \gamma_{\alpha\beta}(t_q)$. Now, since $|u(t)| < M$, $t \geqslant 0$, all relays satisfying $\alpha \geqslant M$ or $\beta \leqslant -M$ will not switch states for all $t \geqslant 0$. Next, consider the Preisach boundary at $t = t_q > t_N$. Since $|u(t)| < \varepsilon$, $t > t_N$, $\gamma_{\alpha\beta}(t)$ satisfying $\alpha \geqslant \varepsilon$ and $\beta \leqslant -\varepsilon$ (i.e., points corresponding to area A in Fig. 4) will remain unchanged for all $t > t_N$. Furthermore, $\gamma_{\alpha\beta}(t)$ satisfying $\beta \geqslant \varepsilon$ (i.e., points corresponding to area C in Fig. 4), and $\alpha \leqslant \varepsilon$ (i.e., points corresponding to area F in Fig. 4) will maintain the value of $-1$ and 1, respectively, for all $t > t_N$. Hence, only points corresponding to areas $B$, $D$, and $E$ shown in Fig. 4 will switch states after time $t_N$. However, since the boundary between $\mathscr{P}_+(t)$ and $\mathscr{P}_-(t)$ has a decreasing staircase shape, the maximum area between $s(t_p, x_h(0), u(t_p))$ and $s(t_q, x_h(0), u(t_q))$ will be either $B \cup E$ or $D \cup E$, whose area is equal to $2\varepsilon M$. $\square$

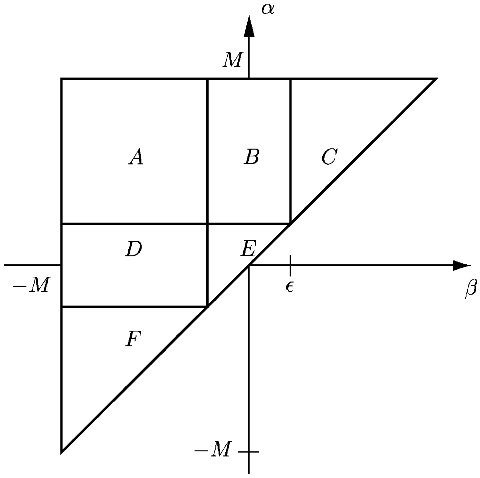


Fig. 4. Regions in $\mathcal{D}$.

**Theorem 3.1.** *Let* $u \in \mathcal{U}$ *be such that* $u(t) \to 0$ *as* $t \to \infty$. *Then there exists* $x_{\rm he} \in \mathcal{D}$ *such that* $\lim_{t\to\infty} s(t, x_{\rm h}(0), u(t)) = x_{\rm he}$ *for all* $x_{\rm h}(0) \in \mathcal{D}$.

**Proof.** Since $u \in \mathcal{U}$, $|u(t)| < M$, $t \geqslant 0$. Now, for $t, t_1 \geqslant 0$ such that $t > t_1$, let $|u(t)| < 1/2M$. Then, it follows from the Proposition 3.2 that $\rho(s(t_p, x_{\rm h}(0), u(t_p)), s(t_q, x_{\rm h}(0), u(t_q))) < 1$, $t_p, t_q > t_1$, $x_{\rm h}(0) \in \mathcal{D}$. Repeating this procedure with $t > t_n > t_{n-1}$, it follows that $\rho(s(t_p, x_{\rm h}(0), u(t_p)), s(t_q, x_{\rm h}(0), u(t_q))) < 1/n$, $t_p, t_q > t_n$. Now, consider the sequence $\{s(t_n, x_{\rm h}(0), u(t_n))\}_{n=1}^{\infty}$. Since $t_n > t_{n-1}$ and $\rho(s(t_p, x_{\rm h}(0), u(t_n)), s(t_q, x_{\rm h}(0), u(t_{n-1}))) \to 0$ as $t_n, t_{n-1} \to \infty$, $\{s(t_n, x_{\rm h}(0), u(t_n))\}_{n=1}^{\infty}$ is a Cauchy sequence. Hence, since by Lemma 3.1 the metric space $(\mathcal{D}, \rho)$ is complete, it follows that $\{s(t_n, x_{\rm h}(0), u(t_n))\}_{n=1}^{\infty}$ is convergent which proves the result. □

Next, we show that the Preisach dynamical system is dissipative with respect to a certain supply rate. To see this, consider a weighted relay $\gamma_{\alpha\beta}(\cdot)$ with generalized weighted power $p(t) = \mu(\alpha, \beta)u(t)\dot{\gamma}_{\alpha,\beta}[u(t)]$ so that

$$\int_{t_1}^{t_2} p(t)\mathrm{d}t = \int_{t_1}^{t_2} \mu(\alpha, \beta)u(t)\dot{\gamma}_{\alpha\beta}[u(t)]\mathrm{d}t, \quad t_2 \geqslant t_1. \tag{14}$$

Since $\gamma_{\alpha\beta}[u(t)]$ is constant and can switch between $\pm 1$ over the time interval $t_1 \leqslant t \leqslant t_2$, it follows that Eq. (14) is equal to either $2\mu(\alpha, \beta)\alpha$ or $-2\mu(\alpha, \beta)\beta$ after a switch, depending on the previous state of $\gamma_{\alpha\beta}(\cdot)$. In addition, since $\mu(\alpha, \beta)$ is by assumption nonnegative, the sign of $\int_{t_1}^{t_2} p(t)\mathrm{d}t$ will depend on $\alpha$, $\beta$ and the previous state of the relay and possibly the number of switchings between $t_1$ and $t_2$. Furthermore, since $p(t)$ is the product of the input and the time rate of change of output of the relay, $p(t)$ can be interpreted as the generalized power of $\gamma_{\alpha\beta}(\cdot)$ and $\int_{t_1}^{t_2} p(t)\mathrm{d}t$ as the generalized energy transfer of $\gamma_{\alpha\beta}(\cdot)$. Noting that negative energy transfer represents the energy recovered from $\gamma_{\alpha\beta}(\cdot)$, we can define the stored, or available, energy of $\gamma_{\alpha\beta}(\cdot)$ as the positive part of the value of $\int_{t_1}^{t_2} p(t)\mathrm{d}t$.

Using the above energy formalism, the concept of stored energy can be extended to the Preisach dynamical model. Specifically, if $x_\mathrm{h}(t)\in\mathscr{D}, t\geqslant 0$, denotes the state of a Preisach model $\sigma_\mathrm{h}(\cdot)$, then the total generalized stored energy of $\sigma_\mathrm{h}(\cdot)$ is given by

$$V_\mathrm{h}(x_\mathrm{h})\triangleq 2\int\int_{\mathscr{Q}_1\cap\mathscr{P}_+(t)} \mu(\alpha,\beta)\beta\,\mathrm{d}\alpha\,\mathrm{d}\beta - 2\int\int_{\mathscr{Q}_3\cap\mathscr{P}_-(t)} \mu(\alpha,\beta)\alpha\,\mathrm{d}\alpha\,\mathrm{d}\beta, \tag{15}$$

where

$$\mathscr{Q}_1\triangleq\{(\alpha,\beta):\ \alpha>0,\beta>0\}\cap\mathscr{P}, \tag{16}$$

$$\mathscr{Q}_3\triangleq\{(\alpha,\beta):\ \alpha<0,\beta<0\}\cap\mathscr{P}. \tag{17}$$

Note that if $\mu(\alpha,\beta)\geqslant 0$, then $V_\mathrm{h}(x_\mathrm{h})\geqslant 0$ for all $x_\mathrm{h}\in\mathscr{D}$. The next result shows that the Preisach dynamical system is dissipative with respect to the supply rate $r(u,y)=2u\dot{y}$. For this result and the reminder of the paper, we assume that $\mu:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is bounded, piecewise continuous, and $\mu(\alpha,\beta)\geqslant 0$ for $\alpha\geqslant\beta$.

**Theorem 3.2** (Gorbet et al. [24]). *Let $u(\cdot)$ and $y(\cdot)$ denote the input and output, respectively, to the Preisach dynamical model $\sigma_\mathrm{h}(\cdot)$. Then there exists a continuous nonnegative-definite function $V_\mathrm{h}:\mathscr{D}\to\mathbb{R}$ such that*

$$V_\mathrm{h}(x_\mathrm{h}(t))\leqslant V_\mathrm{h}(x_\mathrm{h}(t_0)) + \int_{t_0}^{t} 2u(s)\dot{y}(s)\mathrm{d}s, \quad t\geqslant t_0, \tag{18}$$

*for all $t_0, t\geqslant 0$, where $x_\mathrm{h}(t)\in\mathscr{D}$, $t\geqslant 0$, is the state of the Preisach model.*

An excellent exposition of the above passivity formalism of the Preisach model can be found in Ref. [26]. The next result shows that the Preisach model attains a state of minimum stored energy when $u(t)=0$ for a given time $t$.

**Proposition 3.3** (Gorbet et al. [24]). *Let $u(\cdot)$, $x_\mathrm{h}(\cdot)$, and $V_\mathrm{h}:\mathscr{D}\to\mathbb{R}$ denote the input, state, and stored energy, respectively, of the Preisach dynamical model $\sigma_\mathrm{h}(\cdot)$. If $u(t)=0$ for a given time $t\geqslant 0$, then $V_\mathrm{h}(x_\mathrm{h}(t))=0$, $x_\mathrm{h}(t)\in\mathscr{D}$.*

It follows from Proposition 3.3 that if $x_\mathrm{h}(t)\in\mathscr{D}_\mathrm{e}$ at any given time $t$, then $V_\mathrm{h}(x_\mathrm{h}(t))=0$, which implies $\mathscr{D}_\mathrm{e}\subseteq\{x_\mathrm{h}\in\mathscr{D}: V_\mathrm{h}(x_\mathrm{h})=0\}$. Now, suppose there exists a neighborhood of the origin of the Preisach plane $\mathscr{P}$ such that $\mu(\alpha,\beta)>0$, $\alpha\geqslant\beta$. Then every Preisach state $s(t,x_\mathrm{h}(0),u(t))$ with zero stored energy at a given time $t\geqslant 0$ must correspond to the case where $u(t)=0$, since if $u(t)\neq 0$, $V_\mathrm{h}(x_\mathrm{h}(t))\neq 0$ by Equation (15). This implies that $\mathscr{D}_\mathrm{e}\supseteq\{x_\mathrm{h}\in\mathscr{D}: V_\mathrm{h}(x_\mathrm{h})=0\}$. Hence, in the case where $\mu(\alpha,\beta)>0$, $\alpha\geqslant\beta$, in a neighborhood of the origin of the Preisach plane, $\mathscr{D}_\mathrm{e}=\{x_\mathrm{h}\in\mathscr{D}: V_\mathrm{h}(x_\mathrm{h})=0\}$ which gives an alternative characterization of the set of all equilibria of the Preisach model.

## 4. Stability conditions for systems with input hystereses nonlinearities

In this section, we develop absolute stability criteria for feedback systems with hystereses nonlinearities $\sigma_{\rm h} : \mathscr{S}^m \to \mathscr{S}^m$, where $\mathscr{S}^m \triangleq \{u(\cdot) \in {\rm C}^1 : \int_{-\infty}^{\infty} (u^{\rm T}(t)u(t) + \dot{u}^{\rm T}(t)\dot{u}(t)){\rm d}t < \infty\}$ denotes a real separable Sobolev space of $m \times 1$ functions. Specifically, given a finite dimensional dynamical system with state-space realization

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

we derive sufficient conditions that guarantee partial asymptotic stability; that is, asymptotic stability with respect to part of the system states, of the feedback interconnection involving the linear dynamical system $G(s)$ and the feedback multivariable, component decoupled hysteresis nonlinearity $\sigma_{\rm h}(\cdot) \in \Sigma_{\rm h}$, where

$$\Sigma_{\rm h} \triangleq \{\sigma_{\rm h} : \mathscr{S}^m \to \mathscr{S}^m : \sigma_{\rm h}(y(t)) = [\sigma_{{\rm h}_1}(y_1) \ldots \sigma_{{\rm h}_m}(y_m)]^{\rm T}\}. \tag{19}$$

Here, $\sigma_{{\rm h}_i}(\cdot)$, $i \in \{1, \ldots, m\}$, denotes a Preisach-type hysteresis nonlinearity with bounded, piecewise continuous, and nonnegative definite weighting function $\mu_i(\alpha_i, \beta_i)$ such that $\alpha_i \geqslant \beta_i$. Furthermore, we assume that $\mu_i(\alpha_i, \beta_i) > 0$, $\alpha_i \geqslant \beta_i$, in a neighborhood of the origin of the Preisach plane $\mathscr{D}$ for each $\sigma_{{\rm h}_i}(\cdot)$. Note that this additional assumption on $\mu_i(\alpha_i, \beta_i)$ ensures that $\sigma_{{\rm h}_i}(\cdot)$ has a hysteretic effect for arbitrarily small input changes, and hence hysteresis nonlinearities with local memories such as stiction and backlash are excluded from the set $\Sigma_{\rm h}$. However, since Preisach models with the above additional assumption can capture most of the hystereses nonlinearities arising in smart material actuators such as shape memory alloys and piezoceramics [2,3], the class of hystereses nonlinearities $\Sigma_{\rm h}$ is quite general.

Note that the negative feedback interconnection of $G(s)$ and $\sigma_{\rm h}(\cdot)$ is given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geqslant 0, \tag{20}$$

$$y(t) = Cx(t) + Du(t), \tag{21}$$

$$u(t) = -\sigma_{\rm h}(y(t)), \tag{22}$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}^m$, and $\sigma_i(\cdot) \in \Sigma_{\rm h}$. Furthermore, note that (see Fig. 5) the stability of the closed-loop system (20)–(22) is equivalent to the stability of the negative feedback interconnection of $\tilde{G}(s)$ and $\tilde{\sigma}_{\rm h}(\cdot)$, where

$$\tilde{G}(s) \triangleq s^{-1} G(s), \quad \tilde{\sigma}_{\rm h}(y(t)) \triangleq \frac{\rm d}{{\rm d}t} \sigma_{\rm h}(y(t)) = \left[\frac{\rm d}{{\rm d}t} \sigma_{{\rm h}_1}(y_1(t)) \cdots \frac{\rm d}{{\rm d}t} \sigma_{{\rm h}_m}(y_m(t))\right]^{\rm T}. \tag{23}$$

Next, we provide an absolute stability result for the feedback system given by Eqs. (20)–(22) where $\sigma_{\rm h}(\cdot) \in \Sigma_{\rm h}$ is a Preisach multivariable, component decoupled hysteresis nonlinearity. For the statement of this result, define $\tilde{x} \triangleq [x^{\rm T}, x_{\rm h}^*]^* \in \mathbb{R}^n \times \mathscr{D}$, where $(\,)^*$ denotes the adjoint operator on $\mathscr{D}$, and let $\tilde{s}(t, \tilde{x})$ denote the trajectory of the feedback system (20)–(22).

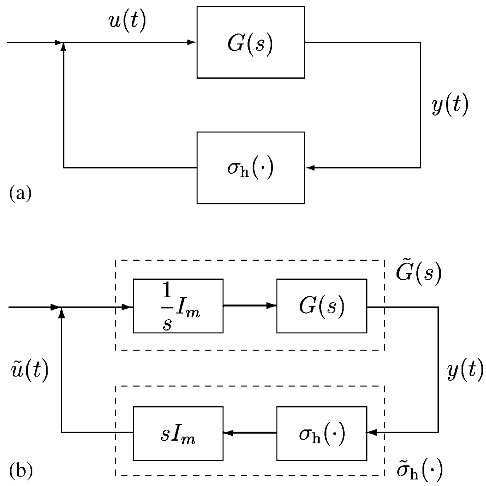


Fig. 5. Equivalent feedback systems.

**Theorem 4.1.** *Let*

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

*be asymptotically stable, assume rank $B = m$, suppose that every element of $G(s)$ has at least one zero at the origin, and assume $s^{-1}G(s)$ is strictly positive real. Then, there exists a neighborhood of the origin $\mathcal{D}_0 \subseteq \mathbb{R}^n \times \mathcal{D}$ such that if $\tilde{x}(0) \in \mathcal{D}_0$, then all equilibria of the feedback system* (20)–(22) *are semistable. Furthermore, $x(t) \to 0$ as $t \to \infty$.*

**Proof.** First note that the stability of the negative feedback interconnection of $G(s)$ and $\sigma_h(\cdot)$ is equivalent to the stability of the negative feedback interconnection of $\tilde{G}(s) = s^{-1}G(s)$ and $\tilde{\sigma}_h(\cdot)$. Now, since

$$s^{-1}G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline CA^{-1} & 0 \end{array}\right]$$

is strictly positive real and $G(0) = 0$, it follows from Theorem 2.2 that there exists matrices $P$ and $L$, with $P$ positive definite, and a positive constant $\varepsilon$ such that Eqs. (7) and (8) hold. Furthermore, since $\sigma_h(\cdot) \in \Sigma_h$, it follows from Theorem 3.2 that $\tilde{\sigma}_h(\cdot)$ is dissipative with respect to the supply rate $y^T(t)\tilde{u}(t)$, where $\tilde{u}(t) = \tilde{\sigma}_h(y(t))$. Hence, there exists a continuous nonnegative-definite storage function $V_h : \mathcal{D} \to \mathbb{R}$ such that

$$V_h(x_h(t)) \leqslant V_h(x_h(0)) + 2\int_0^t y^T(s)\tilde{u}(s)\mathrm{d}t, \quad t \geqslant 0, \tag{24}$$

where $x_h(t) \in \mathcal{D}$, $t \geqslant 0$, is the dynamic state of $\tilde{\sigma}_h(\cdot)$. Next, since $s^{-1}G(s)$ is strictly positive real, $V_p(x) \triangleq x^T P x$, where $P > 0$ satisfies Eqs. (7) and (8), is a candidate storage function for the dynamical system $\tilde{G}(s)$ with input $\tilde{u}(t)$ and output $y(t)$. Hence,

$$\begin{aligned}\dot{V}_{\rm p}(x(t)) &= V_{\rm p}'(x(t))[Ax(t) - B\tilde{u}(t)] \\ &= x^{\rm T}(A^{\rm T}P + PA)x(t) - 2x^{\rm T}(t)PB\tilde{u}(t) \\ &= -\varepsilon x^{\rm T}(t)Px(t) - x^{\rm T}(t)L^{\rm T}Lx(t) - 2x^{\rm T}A^{-{\rm T}}C^{\rm T}\tilde{u}(t) \\ &\leqslant -\varepsilon x^{\rm T}(t)Px(t) - 2y^{\rm T}(t)\tilde{u}(t),\end{aligned} \tag{25}$$

or, equivalently,

$$V_{\rm p}(x(t)) \leqslant V_{\rm p}(x(0)) - \varepsilon \int_0^t x^{\rm T}(s)Px(s){\rm d}s - 2\int_0^t y^{\rm T}(s)\,\tilde{u}(s){\rm d}t, \quad t \geqslant 0. \tag{26}$$

Now, defining $V(x, x_{\rm h}) \triangleq V_{\rm p}(x) + V_{\rm h}(x_{\rm h})$ and adding Eqs. (24) and (26) yields

$$V(x(t), x_{\rm h}(t)) \leqslant V(x(0), x_{\rm h}(0)) - \varepsilon \int_0^t x^{\rm T}(s)Px(s){\rm d}s, \quad t \geqslant 0. \tag{27}$$

Next, it follows from Theorem 2.1 that $\tilde{x}(t) \to \mathcal{M} = \bigcup_{\gamma \in \mathbb{R}} \mathcal{M}_\gamma$ as $t \to \infty$, where $\mathcal{M}_\gamma$ is the largest invariant set contained in $V^{-1}(\gamma)$. Now, for a given $\gamma \in \mathbb{R}$, if $\tilde{x}(0) \in \mathcal{M}_\gamma$, it follows from Eq. (27) that

$$\gamma \leqslant \gamma - \varepsilon \int_0^t x^{\rm T}(s)Px(s){\rm d}s, \quad t \geqslant 0, \tag{28}$$

which implies that $x(t) = 0$ for all $t \geqslant 0$. Since $\gamma$ is arbitrary, $\mathcal{M} \subset \{\tilde{x}(t){:}\ x(t) \equiv 0\}$ and hence $x(t) \to 0$ as $t \to \infty$. Next, since rank $B = m$ and $x(t) \to 0$ as $t \to \infty$, it follows that $u(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$. Furthermore, it follows from Theorem 3.1, with $u(t)$ replaced by $y(t)$, that there exists $x_{\rm he} \in \mathcal{D}$ such that $\lim_{t\to\infty} s(t, x_{\rm h}(0), y(t)) = x_{\rm he}$ for all $x_{\rm h}(0) \in \mathcal{D}$. Now, since $V_{\rm h}(x_{\rm h}) = 0$ if and only if $x_{\rm h} \in \mathcal{D}_{\rm e}$, it follows that $V(x, x_{\rm h})$ defined above is a candidate Lyapunov function for the closed-loop system (20)–(22). Thus, using Eq. (27), it follows that $V(x(t), x_{\rm h}(t)) \leqslant V(x(\tau), x_{\rm h}(\tau))$, $t \geqslant \tau$, and hence Lyapunov stability of the feedback system (20)–(22) follows from standard arguments. Furthermore, semistability of all equilibria of the closed-loop system (20)–(22) follows from the fact that $\lim_{t\to\infty} s(t, x_{\rm h}(0), y(t)) = x_{\rm he}$ and $\lim_{t\to\infty} x(t) = 0$. Finally, asymptotic stability of the zero solution $x(t) \equiv 0$ is immediate. □

**Remark 4.1.** Note that if $V_{\rm h} : \mathcal{D} \to \mathbb{R}$ is uniformly unbounded on $\mathcal{N} \subset \mathcal{D}$; that is, given $\alpha > 0$ there exists a compact set $\mathcal{D}_{\rm c} \subset \mathcal{N}$ with $\mathcal{D}_{\rm c} \neq \mathcal{N}$ such that $V(x_{\rm h}) \geqslant \alpha$, $x_{\rm h} \notin \mathcal{D}_{\rm c}$, then the results of Theorem 4.1 are global.

**Remark 4.2.** If we assume that $\sigma_{{\rm h}_i}(\cdot)$, $i = \{1, \ldots, m\}$, in $\Sigma_{\rm h}$ are such that $\mu_i(\alpha_i, \beta_i)$, $\alpha_i \geqslant \beta_i$, are not positive definite in a neighborhood of the origin, then Theorem 4.1 guarantees semistability of the invariant set $\{x_{\rm h} \in \mathcal{D} : V_{\rm h}(x_{\rm h}) = 0\} \cup \{0\} \supseteq \mathcal{D}_{\rm e} \cup \{0\}$.

## 5. Static output feedback controllers with actuator hystereses nonlinearities

In this section, we introduce a feedback stabilization problem involving input hystereses nonlinearities (see Fig. 6). The goal of this problem is to determine a static output feedback controller that stabilizes a given linear dynamical system with hysteretic actuator nonlinearities $\sigma_{\mathrm{h}}(\cdot) \in \Sigma_{\mathrm{h}}$.

**Output feedback stabilization problem.** Given the $n$th-order controllable and observable plant

$$\dot{x}(t) = Ax(t) - B\sigma_{\mathrm{h}}(u(t)), \quad x(0) = x_0, \quad t \geqslant 0, \tag{29}$$

$$y(t) = Cx(t) - D\sigma_{\mathrm{h}}(u(t)), \tag{30}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, and $\sigma_{\mathrm{h}}(\cdot) \in \Sigma_{\mathrm{h}}$, determine an output feedback controller

$$u(t) = Ky(t), \tag{31}$$

where $K \in \mathbb{R}^{m \times l}$, that satisfies the following design criteria:

(i) the closed-loop system (29)–(31) is semistable; and
(ii) $x(t) \to 0$ as $t \to \infty$.

**Theorem 5.1.** *Consider the feedback system given by* Eqs. (29)–(31). *Let*

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

*be asymptotically stable, assume rank $B = m$, and suppose that every element of $G(s)$ has at least one zero at the origin. Furthermore, assume that there exist matrices $P \in \mathbb{R}^{n \times n}$, $E \in \mathbb{R}^{p \times n}$, and $K \in \mathbb{R}^{m \times l}$, with $P$ positive definite, and a positive constant $\varepsilon$ such that*

$$0 = A^{\mathrm{T}}P + PA + \varepsilon P + E^{\mathrm{T}}E, \tag{32}$$

$$0 = B^{\mathrm{T}}P - KCA^{-1}. \tag{33}$$

*Then there exists a neighborhood of the origin $\mathcal{D}_0 \subseteq \mathbb{R}^n \times \mathcal{D}$ such that if $\tilde{x}(0) \in \mathcal{D}_0$, then all equilibria of the closed-loop system* (29)–(31) *are semistable and $x(t) \to 0$ as $t \to \infty$.*

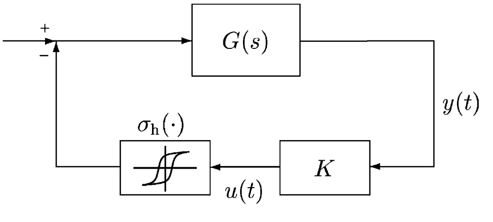


Fig. 6. Feedback system with a hysteresis input nonlinearity.

**Proof.** The result is a direct consequence of Theorem 4.1 by noting that the stability of the closed-loop system (29)–(31) is equivalent to the stability of the negative feedback interconnection of

$$s^{-1}KG(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline KCA^{-1} & 0 \end{array}\right]$$

and $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{\mathrm{h}}(\cdot)$. $\square$

**Remark 5.1.** It is important to emphasize that Theorem 5.1 provides output feedback controllers for open-loop asymptotically stable systems with input hysteresis nonlinearities. Relevant applications include, for example, damped flexible structures controlled by adaptive smart actuator materials exhibiting significant hysteresis.

Note that the controller gain $K$ and the positive definite matrix $P$ in Theorem 5.1 can be computed by considering the set of linear matrix inequalities (LMIs)

$$P > 0, \tag{34}$$

$$\begin{bmatrix} A^{\mathrm{T}}P + PA & E^{\mathrm{T}} \\ E & -I_n \end{bmatrix} < 0, \tag{35}$$

$$\begin{bmatrix} \delta I_m & B^{\mathrm{T}}PA - KC \\ (B^{\mathrm{T}}PA - KC)^{\mathrm{T}} & I_n \end{bmatrix} \geqslant 0, \tag{36}$$

where $\delta \geqslant 0$. Specifically, it can easily be shown using Schur compliments that if there exists $P$, $E$, and $K$ satisfying the above set of LMIs with $\delta = 0$, then $P$ and $K$ satisfying Eqs. (34)–(36) also satisfy Eqs. (32) and (33). Hence, a static output feedback controller can be obtained by considering a minimization problem on $\delta$ subject to the LMIs (34)–(36).

## 6. Illustrative numerical example

In this section, we provide a numerical example to demonstrate the proposed control framework. Specifically, we consider an asymptotically stable linear dynamical system given by Eqs. (29) and (30), where

$$A = \begin{bmatrix} -10 & -10 \\ 1 & 0 \end{bmatrix}, \quad B = I_2, \quad C = \begin{bmatrix} 0 & 1 \\ 5 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -0.1 & -1 \\ 0 & 5 \end{bmatrix},$$

with a multivariable hysteresis nonlinearity $\sigma_{\mathrm{h}}(\cdot) = [\sigma_{\mathrm{h}_1}(\cdot)\sigma_{\mathrm{h}_2}(\cdot)]^{\mathrm{T}} \in \Sigma_{\mathrm{h}}$. Note that $-CA^{-1}B + D = 0$ and hence $G(0) = 0$. Here, we assume uniform weighting functions $\mu_i(\alpha_i, \beta_i) = 0.03$ for all $\alpha_i$, $\beta_i$, $i \in \{1, 2\}$. The input–output Lissajous map of each element of $\sigma_{\mathrm{h}}(\cdot)$ is shown in Fig. 7.

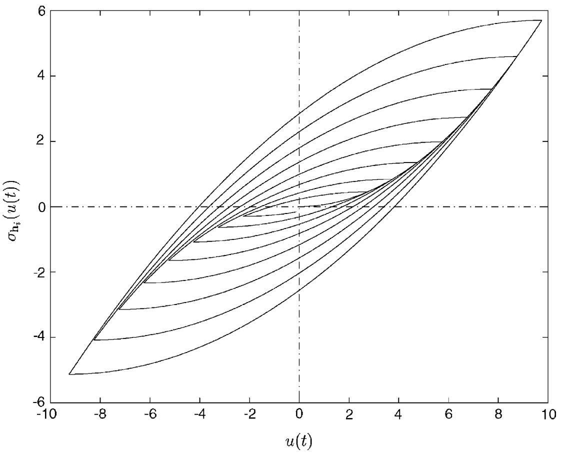


Fig. 7. Inputs–output Lissajous map of $\sigma_{h_i}(\cdot)$.

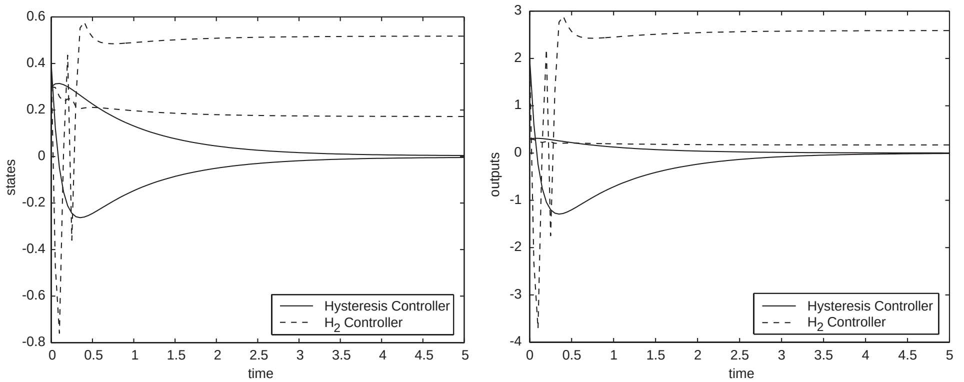


Fig. 8. Comparison of $H_2$-optimal and hysteresis controllers.

Using Theorem 5.1, a static output feedback controller $u(t) = Ky(t)$ was designed for $E = B$ by minimizing $\delta$ subject to the LMIs (34)–(36). The minimum $\delta$ achieved was $2.47 \times 10^{-15}$ indicating that

$$P = \begin{bmatrix} 0.055 & 0.05 \\ 0.05 & 1.05 \end{bmatrix}, \quad K = \begin{bmatrix} -0.55 & -0.1 \\ -0.5 & 0.11 \end{bmatrix},$$

satisfying Eqs. (34)–(36) also satisfy (32) and(33). Fig. 8 compares the output and plant state response with the hysteresis controller designed using Theorem 5.1 and an $H_2$-optimal static output feedback controller [27] for an initial condition $x(0) = [0.4\,0.3]^T$. This comparison illustrates that the $H_2$-optimal controller drives the controlled plant states to a nonzero equilibrium while the proposed hysteresis controller guarantees partial asymptotic stability of the closed-loop system.

## 7. Conclusion

A static output feedback controller analysis and synthesis framework for linear systems with hystereses input nonlinearities was developed. Specifically, by transforming the hystereses nonlinearities into dissipative input–output dynamical operators, dissipativity theory was used to analyze and design linear controllers for systems with hysteretic actuators. The feedback controller guarantees asymptotic stability of the plant states and semistability of the closed-loop system. Finally, a numerical example was presented to demonstrate the effectiveness of the proposed hysteresis control approach.

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