

# Stability margins of nonlinear optimal regulators with nonquadratic performance criteria involving cross-weighting terms<sup>☆</sup>

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## Abstract

In this paper we derive guaranteed gain, sector, and disk margins for nonlinear optimal and inverse optimal regulators that minimize a nonlinear-nonquadratic performance criterion involving cross-weighting terms. Specifically, sufficient conditions that guarantee gain, sector, and disk margins are given in terms of the state, control, and cross-weighting nonlinear-nonquadratic weighting functions. The proposed results provide a generalization of the “meaningful” inverse optimal nonlinear regulator stability margins as well as the classical linear-quadratic optimal regulator gain and phase margins. © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

The gain and phase margins of state feedback linear-quadratic optimal regulators are well known [1,8,12]. In particular, in terms of classical control relative stability notions, these controllers possess at least a  $\pm 60^\circ$  phase margin, infinite gain margin, and 50% gain reduction for each control channel. Alternatively, in terms of absolute stability theory [1] these controllers guarantee sector margins in that the closed-loop system will remain asymptotically stable in the face of a memoryless static input nonlinearity contained in the conic sector  $(\frac{1}{2}, \infty)$ . In

both cases, these results hold if the integrand of the quadratic performance criterion is chosen to be a quadratic nonnegative-definite function of the state and a quadratic positive-definite function of the control with a diagonal weighting matrix. Gain and phase margins of state feedback linear-quadratic optimal regulators involving cross-weighting terms in the quadratic performance criterion were obtained in [2]. Specifically, the authors in [2] provide explicit connections between relative stability margins and the selection of the state, control, and cross-weighting matrices. However, unlike the standard linear-quadratic case, no sector margin guarantees were shown in the linear-quadratic problem with cross-weighting terms.

The problem of guaranteed sector margins for state feedback nonlinear-nonquadratic *inverse* optimal regulators has also been considered in the literature [3,7,10,11]. Specifically, nonlinear Hamilton–Jacobi–Bellman inverse optimal controllers that

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minimize a meaningful (in the terminology of [3,13]) nonlinear-nonquadratic performance criterion involving a nonlinear-nonquadratic, nonnegative-definite function of the state and a quadratic positive-definite function of the feedback control are shown to possess sector margin guarantees to component decoupled input nonlinearities in the conic sector  $(\frac{1}{2}, \infty)$ . These results have been recently extended in [13] to disk margin guarantees where asymptotic stability of the closed-loop system is guaranteed in the face of a dissipative dynamic input operator.

Motivated by the recent results of [4,5] involving inverse optimal backstepping controllers that minimize a derived nonlinear-nonquadratic performance criterion involving a nonlinear-nonquadratic function of the state and a linear and quadratic function of the feedback control, gain, sector, and disk margins are derived for nonlinear optimal regulators with nonquadratic performance criteria involving cross-weighting terms. Specifically, gain, sector, and disk margin guarantees are obtained for nonlinear dynamic systems controlled by nonlinear optimal and inverse optimal Hamilton–Jacobi–Bellman controllers [15] that minimize a nonlinear-nonquadratic performance criterion with cross-weighting terms. In the case where the cross-weighting term in the performance criterion is deleted our results recover the gain, sector, and disk margins of [13]. Alternatively, retaining the cross-terms in the performance criterion and specializing the nonlinear-nonquadratic problem to a linear-quadratic problem our results recover the gain and phase margins of [2]. Finally, we note that even though the inclusion of cross-weighting terms in the performance criterion is shown to degrade gain, sector, and disk margins, the extra flexibility provided by the cross-weighting terms makes it possible to guarantee optimal and inverse optimal nonlinear controllers that may be far superior in terms of transient performance over meaningful inverse optimal controllers.

## 2. Mathematical preliminaries

In this section we establish definitions, notation, and a key result used later in the paper. Let  $\mathbb{R}$  denote the real numbers, let  $(\cdot)^T$  denote transpose, and let  $I_n$  or  $I$  denote the  $n \times n$  identity matrix. Furthermore, we write  $\|\cdot\|$  for the Euclidean vector norm,  $\sigma_{\max}(\cdot)$  for the maximum singular value,  $\sigma_{\min}(\cdot)$  for the minimum singular value, and  $M \geq 0$  (resp.,  $M > 0$ ) to denote

the fact that the Hermitian matrix  $M$  is nonnegative (resp., positive) definite.

In this paper we consider nonlinear systems  $\mathcal{G}$  of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \quad (1)$$

$$y(t) = h(x(t)) + J(x(t))u(t), \quad (2)$$

where  $x \in \mathbb{R}^n$ ,  $u, y \in \mathbb{R}^m$ ,  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $G: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ ,  $h: \mathbb{R}^n \rightarrow \mathbb{R}^m$ , and  $J: \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ . We assume that  $f(\cdot)$ ,  $G(\cdot)$ ,  $h(\cdot)$ , and  $J(\cdot)$  are smooth (at least  $C^1$  mappings) and  $f(\cdot)$  has at least one equilibrium so that, without loss of generality,  $f(0) = 0$  and  $h(0) = 0$ . Furthermore, for the nonlinear system  $\mathcal{G}$  we assume that the required properties for the existence and uniqueness of solutions are satisfied, that is,  $u(\cdot)$  satisfies sufficient regularity conditions such that the system (1) has a unique solution forward in time. For the dynamical system  $\mathcal{G}$  given by (1) and (2), a function  $r: \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$  is called a *supply rate* [16] if it is locally integrable, that is, for all input-output pairs  $u, y \in \mathbb{R}^m$ ,  $r(\cdot, \cdot)$  satisfies  $\int_{t_1}^{t_2} |r(u(s), y(s))| ds < \infty$ ,  $t_1, t_2 \geq 0$ .

**Definition 2.1** (Willems [16]). A system  $\mathcal{G}$  of the form (1), (2) is *dissipative with respect to the supply rate*  $r$  if there exists a  $C^0$  nonnegative-definite function  $V_s: \mathbb{R}^n \rightarrow \mathbb{R}$ , called a *storage function*, such that the *dissipation inequality*

$$V_s(x(t)) \leq V_s(x(t_1)) + \int_{t_1}^t r(u(s), y(s)) ds, \quad (3)$$

is satisfied for all  $t_1, t \geq 0$  and where  $x(t)$ ,  $t \geq 0$ , is the solution to (1) with  $u \in \mathbb{R}^m$ .

Next, we consider feedback interconnections of dissipative dynamical systems. Specifically, consider the nonlinear system  $\mathcal{G}$  given by (1), (2) with the nonlinear feedback system  $\mathcal{G}_c$  given by

$$\begin{aligned} \dot{x}_c(t) &= f_c(x_c(t)) + G_c(u_c(t), x_c(t))u_c(t), \\ x_c(0) &= x_{c0}, \quad t \geq 0, \end{aligned} \quad (4)$$

$$y_c(t) = h_c(u_c(t), x_c(t)) + J_c(u_c(t), x_c(t))u_c(t), \quad (5)$$

where  $x_c \in \mathbb{R}^{n_c}$ ,  $u_c, y_c \in \mathbb{R}^m$ ,  $f_c: \mathbb{R}^{n_c} \rightarrow \mathbb{R}^{n_c}$ ,  $G_c: \mathbb{R}^m \times \mathbb{R}^{n_c} \rightarrow \mathbb{R}^{n_c \times m}$ ,  $h_c: \mathbb{R}^m \times \mathbb{R}^{n_c} \rightarrow \mathbb{R}^m$ , and  $J_c: \mathbb{R}^m \times \mathbb{R}^{n_c} \rightarrow \mathbb{R}^{m \times m}$ . The following lemma gives sufficient conditions for global asymptotic stability of

the negative feedback interconnection of  $\mathcal{G}$  and  $\mathcal{G}_c$ . For the statement of this result recall the definition of zero-state observability given in [6].

**Lemma 2.1** (Hill and Moylan [6]). *Consider the closed-loop system consisting of the nonlinear systems  $\mathcal{G}$  given by (1), (2) and  $\mathcal{G}_c$  given by (4), (5). Let  $a, b, a_c, b_c, \delta \in \mathbb{R}$  be such that  $b > 0$ ,  $a < b$ ,  $0 < 2\delta < b - a$ ,  $a_c = a + \delta$ , and  $b_c = b + \delta$ , and assume  $\mathcal{G}$  and  $\mathcal{G}_c$  are zero-state observable. If  $\mathcal{G}$  is dissipative with respect to the supply rate  $r(u, y) = u^T y + (ab/(a+b))y^T y + (1/(a+b))u^T u$  and has radially unbounded storage function and  $\mathcal{G}_c$  is dissipative with respect to the supply rate  $r(u_c, y_c) = u_c^T y_c - (1/(a_c + b_c))y_c^T y_c - (a_c b_c / (a_c + b_c))u_c^T u_c$  and has radially unbounded storage function, then the negative feedback interconnection of  $\mathcal{G}$  and  $\mathcal{G}_c$  is globally asymptotically stable.*

Now, we consider nonlinear systems  $\mathcal{G}$  of the form given by (1), (2) with  $J(x) \equiv 0$  and  $h(x) = -\phi(x)$ , where  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is such that  $\mathcal{G}$  is asymptotically stable with  $u = -y$ . Furthermore, we assume that the system  $\mathcal{G}$  is zero-state observable. In this case,  $\mathcal{G}$  becomes

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \quad (6)$$

$$y(t) = -\phi(x(t)). \quad (7)$$

Next, we define the robustness margins for  $\mathcal{G}$  given by (6), (7). Specifically, consider the negative feedback interconnection of  $\mathcal{G}$  and  $\Delta(\cdot)$ , where  $\Delta(\cdot)$  is either a linear operator  $\Delta(y) = \Delta y$ , a nonlinear static operator  $\Delta(y) = \sigma(y)$ , or a dynamic operator  $\Delta(\cdot)$ . In this case  $u = -\Delta(y)$ . Furthermore, we assume that in the nominal case,  $\Delta(\cdot) = I(\cdot)$  so that the nominal closed-loop system is asymptotically stable.

**Definition 2.2.** Let  $\alpha, \beta \in \mathbb{R}$  be such that  $\alpha < \beta$ . The nonlinear system  $\mathcal{G}$  given by (6), (7) is said to have a *gain margin*  $(\alpha, \beta)$  if the negative feedback interconnection of  $\mathcal{G}$  and  $\Delta(y) = \Delta y$  is globally asymptotically stable for all  $\Delta = \text{diag}(k_1, \dots, k_m)$ , where  $k_i \in (\alpha, \beta)$ ,  $i = 1, \dots, m$ . The nonlinear system  $\mathcal{G}$  given by (6), (7) is said to have a *sector margin*  $(\alpha, \beta)$  if the negative feedback interconnection of  $\mathcal{G}$  and  $\Delta(y) = \sigma(y)$  is globally asymptotically stable for all nonlinearities  $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^m$  such that  $\sigma(0) = 0$ ,  $\sigma(y) = [\sigma_1(y_1) \cdots \sigma_m(y_m)]^T$ , and  $\alpha y_i^2 < \sigma_i(y_i) y_i < \beta y_i^2$ , for all  $y_i \neq 0$ ,  $i = 1, \dots, m$ . The nonlinear system  $\mathcal{G}$  given by (6), (7) is said to

have a *disk margin*  $(\alpha, \beta)$  if the negative feedback interconnection of  $\mathcal{G}$  and  $\Delta(\cdot)$  is globally asymptotically stable for all dynamic operators  $\Delta(\cdot)$  such that  $\Delta(\cdot)$  is zero-state detectable and dissipative with respect to a supply rate  $r(u, y) = u^T y - (1/(\hat{\alpha} + \hat{\beta}))y^T y - (\hat{\alpha}\hat{\beta}/(\hat{\alpha} + \hat{\beta}))u^T u$ , where  $\hat{\alpha} = \alpha + \delta$ ,  $\hat{\beta} = \beta - \delta$ , and  $\delta \in \mathbb{R}$  such that  $0 < 2\delta < \beta - \alpha$ .

**Remark 2.1.** Note that if  $\mathcal{G}$  has a disk margin  $(\alpha, \beta)$  then it has a gain and sector margins  $(\alpha, \beta)$ . Letting  $\beta \rightarrow \infty$  in the definition for disk margin, we recover the definition of disk margins given in [13].

### 3. Gain, sector, and disk margins of nonlinear-nonquadratic optimal regulators

In this section we derive the robustness margins for a nonlinear optimal regulator that minimizes a nonlinear-nonquadratic performance criterion involving a nonlinear-nonquadratic function of the state and a linear and quadratic function of the feedback control. Specifically, we consider the nonlinear system given by (6) with a nonlinear-nonquadratic performance criterion

$$J(x_0, u(\cdot)) \triangleq \int_0^\infty [L_1(x(t)) + L_2(x(t))u(t) + u^T(t)R_2(x(t))u(t)] dt, \quad (8)$$

where  $L_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $L_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{1 \times m}$ , and  $R_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$  are given such that  $R_2(x) > 0$ ,  $x \in \mathbb{R}^n$ , and  $L_2(0) = 0$ . The optimal nonlinear feedback controller  $u = \phi(x)$  that minimizes the nonlinear-nonquadratic performance criterion (8) is given in [15]. For the statement of the next result define the set of asymptotically stabilizing controllers

$$\mathcal{S}(x_0) \triangleq \{u(\cdot) : u \in \mathbb{R}^m \text{ and } x(\cdot) \text{ given by (6) satisfies } x(t) \rightarrow 0 \text{ as } t \rightarrow \infty\},$$

where  $x_0 \in \mathbb{R}^n$ , and define  $\bar{\gamma} \triangleq \sup_{x \in \mathbb{R}^n} \sigma_{\max}(R_2(x))$  and  $\underline{\gamma} \triangleq \inf_{x \in \mathbb{R}^n} \sigma_{\min}(R_2(x))$ , where  $R_2(x)$ ,  $x \in \mathbb{R}^n$ , is such that  $\bar{\gamma} < \infty$  and  $\underline{\gamma} > 0$ .

**Theorem 3.1.** *Consider the nonlinear system  $\mathcal{G}$  given by (6), (7). Assume there exists a  $C^1$  function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  such that*

$$V(0) = 0, \quad V(x) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \quad (9)$$

$$\begin{aligned} & V'(x)[f(x) - \frac{1}{2}G(x)R_2^{-1}(x)L_2^T(x) \\ & - \frac{1}{2}G(x)R_2^{-1}(x)G^T(x)V^T(x)] < 0, \\ & x \in \mathbb{R}^n, x \neq 0, \end{aligned} \quad (10)$$

$$\begin{aligned} 0 &= L_1(x) + V'(x)f(x) - \frac{1}{4}[V'(x)G(x) + L_2(x)] \\ & \times R_2^{-1}(x)[V'(x)G(x) + L_2(x)]^T, \quad x \in \mathbb{R}^n, \end{aligned} \quad (11)$$

and  $V(x) \rightarrow \infty$  as  $\|x\| \rightarrow \infty$ . If there exists  $\theta \in \mathbb{R}$  such that  $0 < \theta < 1$  and

$$(1 - \theta^2)L_1(x) - \frac{1}{4}L_2(x)R_2^{-1}(x)L_2^T(x) \geq 0, \quad x \in \mathbb{R}^n, \quad (12)$$

is satisfied, then, with the feedback control law  $\phi(x) = -\frac{1}{2}R_2^{-1}(x)[V'(x)G(x) + L_2(x)]^T$ , the nonlinear system  $\mathcal{G}$  has a disk margin  $(1/(1 + \eta\theta), 1/(1 - \eta\theta))$ , where  $\eta \triangleq \sqrt{\gamma/\bar{\gamma}}$ . Furthermore, the performance functional (8) minimized in the sense that

$$\begin{aligned} J(x_0, \phi(x(\cdot))) &= \min_{u(\cdot) \in \mathcal{S}(x_0)} J(x_0, u(\cdot)) = V(x_0), \\ & x_0 \in \mathbb{R}^n. \end{aligned} \quad (13)$$

**Proof.** Note that for all  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ , it follows from (11) and (12) that

$$\begin{aligned} & \theta^2 u^T R_2(x) u \\ & \leq \theta^2 u^T R_2(x) u + \left[ \frac{1}{2\sqrt{1 - \theta^2}} L_2(x) R_2^{-1}(x) \right. \\ & \quad \left. + \sqrt{1 - \theta^2} u^T \right] R_2(x) \\ & \quad \times \left[ \frac{1}{2\sqrt{1 - \theta^2}} L_2(x) R_2^{-1}(x) \right. \\ & \quad \left. + \sqrt{1 - \theta^2} u^T \right]^T \\ & = u^T R_2(x) u + \frac{1}{4(1 - \theta^2)} L_2(x) \\ & \quad \times R_2^{-1}(x) L_2^T(x) + L_2(x) u \\ & \leq u^T R_2(x) u + L_2(x) u + L_1(x) \\ & = u^T R_2(x) u + L_2(x) u - V'(x) f(x) \\ & \quad + \phi^T(x) R_2(x) \phi(x) \\ & = [u + y]^T R_2(x) [u + y] \\ & \quad - V'(x) f(x) - V'(x) G(x) u, \end{aligned}$$

which implies that, for all  $u(\cdot) \in \mathbb{R}^m$  and  $t \geq 0$ ,

$$\begin{aligned} \theta^2 u^T(t) R_2(x(t)) u(t) &\leq [u(t) + y(t)]^T R_2(x(t)) \\ & \quad \times [u(t) + y(t)] - \dot{V}(x(t)). \end{aligned}$$

Now, for all  $t_2 > t_1 \geq 0$ , integrating over  $[t_1, t_2]$  yields

$$\begin{aligned} & V(x(t_2)) - V(x(t_1)) \\ & \leq \int_{t_1}^{t_2} \{ [u(t) + y(t)]^T R_2(x(t)) [u(t) + y(t)] \\ & \quad - \theta^2 u^T(t) R_2(x(t)) u(t) \} dt, \end{aligned}$$

which implies that

$$\begin{aligned} & V(x(t_2)) - V(x(t_1)) \\ & \leq \int_{t_1}^{t_2} \{ \bar{\gamma} [u(t) + y(t)]^T [u(t) + y(t)] \\ & \quad - \underline{\gamma} \theta^2 u^T(t) u(t) \} dt. \end{aligned}$$

Hence, with the storage function  $V_s(x) = (1/2\gamma)V(x)$ ,  $\mathcal{G}$  is dissipative with respect to supply rate  $r(u, y) = u^T y + \frac{1}{2}(1 - \eta^2 \theta^2) u^T u + y^T y$ . Now, the result follows immediately from Lemma 2.1 and Definition 2.2 with  $\alpha = 1/(1 + \eta\theta)$  and  $\beta = 1/(1 - \eta\theta)$ . Finally, (13) is a direct consequence of Corollary 2.3 of [15].  $\square$

Next, we specialize the result of Theorem 3.1 to the case where  $L_2(x) \equiv 0$  and  $L_1(x) \geq 0$ ,  $x \in \mathbb{R}^n$ .

**Corollary 3.1.** Consider the nonlinear system  $\mathcal{G}$  given by (6), (7) where  $\phi(x) = -\frac{1}{2}R_2^{-1}(x) \cdot [V'(x)G(x)]^T$  and where  $V(x)$  satisfies (9)–(11). Furthermore, assume  $L_1(x) \geq 0$ ,  $x \in \mathbb{R}^n$ . Then the nonlinear system  $\mathcal{G}$  has a disk margin  $(1/(1 + \eta), 1/(1 - \eta))$ .

**Proof.** The result is a direct consequence of Theorem 3.1. Specifically, if  $L_1(x) \geq 0$ ,  $x \in \mathbb{R}^n$ , and  $L_2(x) \equiv 0$  then (12) is trivially satisfied for all  $\theta \in (0, 1)$ . Now the result follows immediately by letting  $\theta \rightarrow 1$ .  $\square$

**Remark 3.1.** It should be noted that in the case where  $R_2(x) \equiv I$  it follows that  $\eta = 1$  and Corollary 3.1 specializes to the results given in [13]. In this case,  $\mathcal{G}$  has a disk margin of  $(\frac{1}{2}, \infty)$ .

Next, we provide an alternative result that guarantees sector and gain margins for the case in which  $R_2(x)$ ,  $x \in \mathbb{R}^n$ , is diagonal.

**Theorem 3.2.** Consider the nonlinear system  $\mathcal{G}$  given by (6), (7) where  $\phi(x) = -\frac{1}{2}R_2^{-1}(x) \cdot [V'(x)G(x) + L_2(x)]^T$  and where  $V(x)$  satisfies (9)–(11). Furthermore, let  $R_2(x) = \text{diag}(r_1(x), \dots, r_m(x))$ , where  $r_i: \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $r_i(x) > 0$ ,  $i = 1, \dots, m$ . If there exists  $\theta \in \mathbb{R}$  such that  $0 < \theta < 1$  and (12) is satisfied,

then the nonlinear system  $\mathcal{G}$  has a sector (and hence gain) margin  $((1/(1 + \theta), 1/(1 - \theta)))$ .

**Proof.** Let  $\Delta(y) = \sigma(y)$ , where  $\sigma: \mathbb{R}^m \rightarrow \mathbb{R}^m$  is a static nonlinearity such that  $\sigma(0) = 0$ ,  $\sigma(y) = [\sigma_1(y_1) \cdots \sigma_m(y_m)]^T$ , and  $\alpha y_i^2 < \sigma_i(y_i)y_i < \beta y_i^2$ , for all  $y_i \neq 0$ ,  $i = 1, \dots, m$ , where  $\alpha = 1/(1 + \theta)$  and  $\beta = 1/(1 - \theta)$ ; or, equivalently,  $(\sigma_i(y_i) - \alpha y_i)(\sigma_i(y_i) - \beta y_i) < 0$ , for all  $y_i \neq 0$ ,  $i = 1, \dots, m$ . In this case, the closed-loop system (6), (7) with  $u = -\sigma(y)$  is given by

$$\begin{aligned} \dot{x}(t) &= f(x(t)) - G(x(t))\sigma(-\phi(x(t))), \\ x(0) &= x_0, \quad t \geq 0. \end{aligned} \tag{14}$$

Next, consider the Lyapunov function candidate  $V(x)$  satisfying (11) and let  $\dot{V}(x)$  denote the Lyapunov derivative along the trajectories of the closed-loop system (14). Now, it follows from (11) and (12) that

$$\begin{aligned} \dot{V}(x) &= V'(x)f(x) - V'(x)G(x)\sigma(-\phi(x)) \\ &\leq V'(x)f(x) - V'(x)G(x)\sigma(-\phi(x)) + L_1(x) \\ &\quad - \frac{1}{4(1 - \theta^2)}L_2(x)R_2^{-1}(x)L_2^T(x) + (1 - \theta^2) \\ &\quad \times \left[ \sigma(-\phi(x)) - \frac{1}{2(1 - \theta^2)}R_2^{-1}(x)L_2^T(x) \right]^T \\ &\quad \times R_2(x) \left[ \sigma(-\phi(x)) - \frac{1}{2(1 - \theta^2)}R_2^{-1}(x)L_2^T(x) \right] \\ &= V'(x)f(x) + L_1(x) - V'(x)G(x)\sigma(-\phi(x)) \\ &\quad + (1 - \theta^2)\sigma^T(-\phi(x))R_2(x)\sigma(-\phi(x)) \\ &\quad - L_2(x)\sigma(-\phi(x)) \\ &= \phi^T(x)R_2(x)\phi(x) + 2\phi^T(x)R_2(x)\sigma(-\phi(x)) \\ &\quad + (1 - \theta^2)\sigma^T(-\phi(x))R_2(x)\sigma(-\phi(x)) \\ &= \sum_{i=1}^m r_i(x) \left( \frac{1}{\beta} \sigma_i(y_i) - y_i \right) \left( \frac{1}{\alpha} \sigma_i(y_i) - y_i \right) \\ &= \frac{1}{\alpha\beta} \sum_{i=1}^m r_i(x)(\sigma_i(y_i) - \alpha y_i)(\sigma_i(y_i) - \beta y_i) \\ &\leq 0, \end{aligned}$$

which implies that the closed-loop system (14) is Lyapunov stable. Next, let  $\mathcal{R} \triangleq \{x \in \mathbb{R}^n: \dot{V}(x) = 0\}$  and note that  $\dot{V}(x) = 0$  if and only if  $y = 0$ . Now, since  $\mathcal{G}$  is zero-state observable it follows that  $\mathcal{M} \triangleq \{x \in \mathbb{R}^n: x = 0\}$  is the largest invariant set contained in  $\mathcal{R}$ . Hence, it follows from the Krasovskii–LaSalle invariant set theorem [14] that  $x(t) \rightarrow \mathcal{M} = \{0\}$  as  $t \rightarrow \infty$ . Hence, the closed-loop system (14) is globally asymptotically stable for all  $\sigma(\cdot)$  such that

$\alpha y_i^2 < \sigma_i(y_i)y_i < \beta y_i^2$ ,  $y_i \neq 0$ ,  $i = 1, \dots, m$ , which implies that the nonlinear system  $\mathcal{G}$  given by (6), (7) has sector (and hence gain) margins  $(\alpha, \beta)$ .  $\square$

**Remark 3.2.** Note that in the case where  $R_2(x)$ ,  $x \in \mathbb{R}^n$ , is diagonal, Theorem 3.2 guarantees larger gain and sector margins to the gain and sector margin guarantees provided by Theorem 3.1. However, Theorem 3.2 does not provide disk margin guarantees.

**Remark 3.3.** Setting  $L_2(x) \equiv 0$  and requiring that  $L_1(x) \geq 0$ ,  $x \in \mathbb{R}^n$ , specializes Theorem 3.2 to the results given in [13]. Specifically, in this case, Condition (12) is satisfied for all  $\theta \in (0, 1)$  and hence the nonlinear system  $\mathcal{G}$  given by (6), (7) has gain and sector margins of  $(\frac{1}{2}, \infty)$ .

**Remark 3.4.** To specialize Theorem 3.1 to the case of linear systems set  $f(x) = Ax$ ,  $G(x) = B$ ,  $L_1(x) = x^T R_1 x$ ,  $L_2(x) = 2x^T R_{12}$ ,  $R_2(x) = R_2$ , and  $\phi(x) = Kx$ . Now, assuming that  $(A, K)$  is detectable,  $(A, R_1)$  is observable, and  $\sigma_{\max}^2(R_{12}) < \sigma_{\min}(R_1)\sigma_{\min}(R_2)$ , it follows from Theorem 3.1 with  $V(x) = x^T P x$ , where  $P > 0$  is the solution to the algebraic regulator Riccati equation given by

$$\begin{aligned} 0 &= (A - BR_2^{-1}R_{12}^T)^T P + P(A - BR_2^{-1}R_{12}^T) \\ &\quad + R_1 - R_{12}R_2^{-1}R_{12}^T - PBR_2^{-1}B^T P, \end{aligned} \tag{15}$$

that the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{16}$$

$$y(t) = -Kx(t), \tag{17}$$

where  $K = -R_2^{-1}(B^T P + R_{12})$ , has disk margin (and hence sector and gain margins)  $(1/(1 + \eta\theta), 1/(1 - \eta\theta))$ , where

$$\eta = \frac{\sigma_{\min}(R_2)}{\sigma_{\max}(R_2)}, \quad \theta = \left( 1 - \frac{\sigma_{\max}^2(R_{12})}{\sigma_{\min}(R_1)\sigma_{\min}(R_2)} \right)^{1/2}. \tag{18}$$

Furthermore, in the case where  $R_2$  is diagonal, it follows from Theorem 3.2 that the linear system (16), (17) has sector (and hence gain) margin  $(1/(1 + \theta), 1/(1 - \theta))$ , which are precisely the gain margins given in [2] for linear-quadratic optimal regulators with cross-weighting terms in the performance criterion.

**Remark 3.5.** Setting  $R_{12} = 0$  and requiring that  $R_1 \geq 0$  and  $R_2 = \text{diag}(r_1, \dots, r_m)$ , it follows from Remark 3.4

that the linear system given by (16), (17) has a gain and sector margin of  $(\frac{1}{2}, \infty)$ .

**Remark 3.6.** Note that if the linear system given by (16), (17) has disk margins of  $(1/(1+\eta\theta), 1/(1-\eta\theta))$ , it follows that the linear system has a phase margin  $\phi$  given by [9]  $\cos(\phi) = 1 - \eta^2\theta^2/2$ , or, equivalently,  $\sin(\phi/2) = \eta\theta/2$ . Furthermore, in the case where  $R_{12} = 0$  and  $R_2 = I$ , it follows from (18) that  $\theta = \eta = 1$  and hence the linear system given by (16), (17) has a phase margin of  $\pm 60^\circ$  in each input–output channel.

#### 4. Inverse optimality of nonlinear regulators

In this section we give sufficient conditions that guarantee that a given nonlinear feedback controller has pre-specified disk, sector, and gain margins. The first result presents disk margin guarantees.

**Theorem 4.1.** *Let  $\theta \in (0, 1)$  be given. Consider the nonlinear system  $\mathcal{G}$  given by (6), (7) where  $\phi(x)$  is a stabilizing feedback control law. Assume there exist functions  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $R_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$  such that  $R_2(x) > 0$ ,  $x \in \mathbb{R}^n$ , and  $V(\cdot)$  is  $C^1$  and satisfies (9),*

$$V'(x)f(x) + V'(x)G(x)\phi(x) < 0, \quad x \in \mathbb{R}^n, x \neq 0, \quad (19)$$

$$\begin{aligned} & V'(x)f(x) - \phi^T(x)R_2^{-1}(x)\phi(x) \\ & + \frac{1}{1-\theta^2}(\phi^T(x) + \frac{1}{2}V'(x)G(x)R_2^{-1}(x)) \\ & \times R_2(x)(\phi^T(x) + \frac{1}{2}V'(x)G(x)R_2^{-1}(x))^T \leq 0, \\ & x \in \mathbb{R}^n, \end{aligned} \quad (20)$$

and  $V(x) \rightarrow \infty$  as  $\|x\| \rightarrow \infty$ . Then the nonlinear system  $\mathcal{G}$  has a disk margin  $(1/(1+\eta\theta), 1/(1-\eta\theta))$ . Furthermore, with the feedback control law  $\phi(x)$  the performance functional

$$\begin{aligned} J(x_0, u(\cdot)) = & \int_0^\infty [-V'(x(t))(f(x(t))) \\ & + G(x(t))u(t)] \\ & + (\phi(x(t)) - u(t))^T R_2(x(t)) \\ & \times (\phi(x(t)) - u(t))] dt, \end{aligned} \quad (21)$$

is minimized in the sense of (13).

**Proof.** The result is direct consequence of Theorem 3.1 with  $L_1(x) = -V'(x)f(x) + \phi^T(x)R_2(x)\phi(x)$  and  $L_2(x) = -(2\phi^T(x)R_2(x) + V'(x)G(x))$ . Specifically,

in this case, all the conditions of Theorem 3.1 are trivially satisfied. Next, note that (20) is equivalent to (12). The result is now immediate.  $\square$

The next result provides sufficient conditions that guarantee that a given nonlinear feedback controller has pre-specified gain and sector margins.

**Theorem 4.2.** *Let  $\theta \in (0, 1)$  be given. Consider the nonlinear system  $\mathcal{G}$  given by (6), (7) where  $\phi(x)$  is a stabilizing feedback control law. Assume there exist functions  $R_2(x) = \text{diag}(r_1(x), \dots, r_m(x))$ , where  $r_i : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $r_i(x) > 0$ ,  $i = 1, \dots, m$ , and  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $V(\cdot)$  is  $C^1$  and satisfies (9), (19), and (20). Then the nonlinear system  $\mathcal{G}$  has a disk margin  $(1/(1+\theta), 1/(1-\theta))$ . Furthermore, with the feedback control law  $\phi(x)$  the performance functional (21) is minimized in the sense of (13).*

**Proof.** The result is direct consequence of Theorems 3.2 and 4.1.  $\square$

#### 5. Comparison of inverse optimal controllers with meaningful cost functionals and functionals involving cross-weighting terms

In this section we provide a numerical example that demonstrates the extra flexibility provided by inverse optimal controllers with performance criteria involving cross-weighting terms over inverse optimal controllers with meaningful cost functionals. Specifically, consider the controlled Lorentz dynamical system adapted from [15] given by

$$\dot{x}_1(t) = -\sigma x_1(t) + \sigma x_2(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \quad (22)$$

$$\begin{aligned} \dot{x}_2(t) &= r x_1(t) - x_2(t) - x_1(t)x_3(t) + u(t), \\ x_2(0) &= x_{20}, \end{aligned} \quad (23)$$

$$\dot{x}_3(t) = x_1(t)x_2(t) - b x_3(t), \quad x_3(0) = x_{30}, \quad (24)$$

where  $\sigma, r, b > 0$ . Our objective is to design and compare inverse optimal and meaningful inverse optimal controllers that stabilize the origin of the Lorentz equations (22)–(24). Specifically, we compare the guaranteed gain and sector margins to input saturation-type nonlinearities and the transient performance in terms of maximum overshoot of both designs. In order to design an inverse optimal control law for the nonlinear system (22)–(24) consider the Lyapunov function

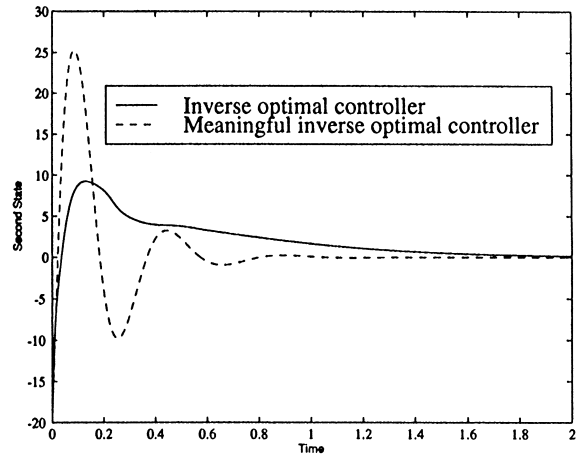
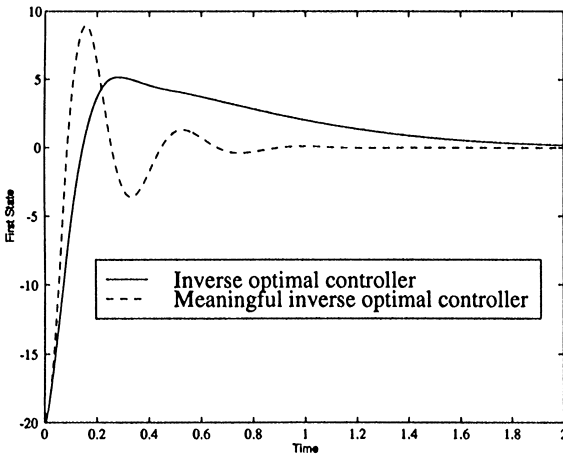


Fig. 1. First and second states versus time.

candidate given by [15]  $V(x) = p_1x_1^2 + p_2x_2^2 + p_2x_3^2$ , where  $x \triangleq [x_1 \ x_2 \ x_3]^T$  and  $p_1, p_2 > 0$ . Next, note that (22)–(24) can be written in the form of (6) with

$$f(x) = \begin{bmatrix} -\sigma x_1 + \sigma x_2 \\ rx_1 - x_2 - x_1x_3 \\ x_1x_2 - bx_3 \end{bmatrix}, \quad G(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \quad (25)$$

Here, we consider the inverse optimal controller given in [15] with performance functional  $L(x, u) = L_1(x) + L_2(x)u + R_2u^2$ , where  $R_2 > 0$ . Now, as in [15], choose  $L_2(x) = (R_2/p_2)(2p_1\sigma + 2p_2r)x_1 - 2p_2x_2$  so that

$$V'(x)[f(x) - \frac{1}{2}G(x)R_2^{-1}(x)L_2^T(x) - \frac{1}{2}G(x)R_2^{-1}G^T(x)V'^T(x)] \quad (26)$$

$$= -2(\sigma p_1x_1^2 + p_2x_2^2 + p_2bx_3^2) < 0, \quad (x_1, x_2, x_3) \neq (0, 0, 0), \quad (27)$$

and hence (10) is satisfied. Next, it follows from Theorem 3.1 that, with

$$L_1(x) = \left[ 2\sigma p_1 + R_2 \left( \frac{p_1}{p_2}\sigma + r \right)^2 \right] x_1^2 - 2(p_1\sigma + p_2r)x_1x_2 + 2p_2x_2^2 + 2p_2bx_3^2, \quad (28)$$

the control law  $\phi(x) = -((p_1/p_2)\sigma + r)x_1$  stabilizes the nonlinear system (22)–(24) while minimizing the

performance functional (8). In this case, (12) becomes  $(1 - \theta^2)[\hat{x}^T M \hat{x} + 2p_2bx_3^2] - \hat{x}^T y y^T \hat{x} \geq 0$ ,  $(29)$

where

$$\hat{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad M \triangleq R \begin{bmatrix} 2\sigma p_1 + R(\frac{p_1}{p_2}\sigma + r)^2 & -(p_1\sigma + p_2r) \\ -(p_1\sigma + p_2r) & 2p_2 \end{bmatrix}, \quad y \triangleq \begin{bmatrix} R(\frac{p_1}{p_2}\sigma + r) \\ -p_2 \end{bmatrix}. \quad (30)$$

Since  $2p_2bx_3^2$  is nonnegative for all  $\theta \in (0, 1)$  it follows that (29) is equivalent to  $(1 - \theta^2)M \geq y y^T$ , which implies that, if  $M$  is invertible, (12) is satisfied for all  $\theta \in (0, 1)$  such that  $1 - \theta^2 \geq y^T M^{-1} y$ . Hence, the maximum possible  $\theta$  such that (12) holds is given by

$$\theta = \sqrt{1 - y^T M^{-1} y}. \quad (31)$$

Now, it follows from Theorem 3.1 that for all  $p_1, p_2, R > 0$ , such that  $\det M \neq 0$ , with  $\phi(x)$  given above the nonlinear system (22)–(24) has disk margins of  $(1/(1 + \theta), 1/(1 - \theta))$ , where  $\theta$  is given by (31). Furthermore, for given  $p_1, p_2, R > 0$  these disk margins are the maximum possible disk margins that are guaranteed by Theorem 3.1. Next, we vary  $p_1, p_2$ , and  $R$  such that  $\theta$  given by (31) is maximized. It can be shown that the maximum is achieved at  $p_1/p_2 = r/\sigma$  and  $p_2/R = 0$  so that  $\theta_{\max} = 1/\sqrt{r + 1}$ . In this case, the control law  $\phi(x)$  is given by  $\phi(x) = -2rx_1$ .

Next, using the control Lyapunov function  $V(x) = p_1x_1^2 + p_2x_2^2 + p_2x_3^2$ , where  $p_1, p_2 > 0$  are such that

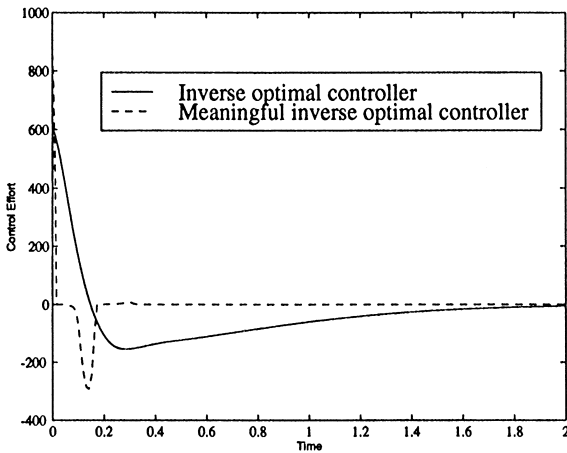


Fig. 2. Control effort versus time.

$p_1/p_2 = r/\sigma$ , we design a meaningful inverse optimal controller using Sontag's universal formula [13] given by

$$\phi(x) = \begin{cases} -(c_0 + \frac{a(x) + \sqrt{a^2(x) + b^4(x)}}{b^2(x)})b(x), & b(x) \neq 0, \\ 0, & b(x) = 0, \end{cases} \quad (32)$$

where  $c_0 \geq 0$ ,  $a(x) \triangleq V'(x)f(x)$ , and  $b(x) \triangleq V'(x)G(x)$ .

Using the initial conditions  $(x_{10}, x_{20}, x_{30}) = (-20, -20, 30)$ , the data parameters  $\sigma = 10$ ,  $r = 15$ ,  $b = 8/3$ , and design parameters  $p_1 = 1.5$  and  $p_2 = 1$  the inverse optimal controller  $\phi(x) = -2rx_1$  and the meaningful inverse optimal controller given by (32) were used to compare closed-loop system performance. First, we note that the downside disk and sector margins of the inverse optimal controller is 0.8 while the meaningful inverse optimal controller guarantees the standard 0.5 downside sector margin with *no* disk margin guarantees. Hence, both controllers have guaranteed robustness sector margins to actuator saturation nonlinearities with, as expected, the meaningful inverse optimal controller having a slightly larger guarantee. However, as shown in Fig. 1, the inverse optimal controller with a cross-weighting term in the performance functional has better transient performance in terms of peak overshoot over the meaningful inverse optimal controller. (We note that the controlled third state is not shown since it is virtually identical for both designs.) Finally, Fig. 2 compares the control effort versus time for both controllers.

## 6. Conclusion

Sufficient conditions for gain, sector, and disk margin guarantees for nonlinear systems controlled by nonlinear optimal and inverse optimal regulators that minimize a nonlinear-nonquadratic performance criterion involving cross-weighting terms were derived. These conditions were shown to provide a generalization of the meaningful inverse optimal nonlinear regulator stability margins as well as the classical linear-quadratic optimal regulator gain and phase margins.

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