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Robust stabilization for systems with parametric uncertainty and time delay

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Abstract

In this paper we use the parameter-dependent Lyapunov function framework developed by Haddad and Bernstein to address the problem of robust stabilization for systems with parametric uncertainty and system delay. The principal result involves the construction of a modified Riccati equation for characterizing a memoryless (delay-independent) feedback controller that guarantees robust stability in the face of parametric uncertainty and time delay. © 1999 The Franklin Institute. Published by Elsevier Science Ltd.

Keywords: Time delay; Robust stabilization; Parametric uncertainty; Parameterized Lyapunov functions

Notation

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$	real numbers, $r\times s$ real matrices, $\mathbb{R}^{r\times 1}$
$(\ )^T$, $(\ )^{-1}$	transpose, inverse
I_r	$r\times r$ identity matrix
$\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$	$r\times r$ symmetric, nonnegative-definite, positive-definite matrices
$Z_1 \leqslant Z_2, Z_1 < Z_2$	$Z_2 - Z_1 \in \mathbb{N}^r$, $Z_2 - Z_1 \in \mathbb{P}^r$; Z_1, $Z_2 \in \mathbb{S}^r$
n, m, m_0	positive integers
x, u	n-, m-dimensional vectors
A, ΔA; B	$n\times n$; $n\times m$ matrices

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A_d, ΔA_d	$n\times n$ matrices
K	$m\times n$ matrix
R_1, R_2	$n\times n, m\times m$ matrices

1. Introduction

A fundamental problem in control engineering is the design of feedback controllers that are insensitive to errors in the control design model. To account for parametric uncertainty in the system dynamics, robust control design methodologies predicated on quadratic stabilizability have been developed in the literature [1, 2]. However, since quadratic stabilizability is equivalent to the existence of a *single* quadratic Lyapunov function for guaranteeing robust stability for a class of real matrix perturbations and since the existence of a single Lyapunov function for each such perturbation is equivalent to a small gain condition, the resulting feedback controllers can be overly conservative for real parameter uncertainty. In a recent series of papers [3–5] a parameter-dependent Lyapunov function framework was developed to address the problem of constant real parameter uncertainty for robust controller synthesis. For robust stability, the form of the parameterized Lyapunov function is critical because the presence of the uncertainty within the Lyapunov function does not allow the uncertain parameters to be arbitrarily time-varying, which renders it less conservative for constant real parameter uncertainty than a fixed quadratic Lyapunov function.

In many applications of feedback control time delays arise frequently in practice. The presence of time delays in a system can severely degrade the closed-loop system performance and can drive the system to instability. Hence, as in the case of system uncertainty, it is of paramount importance that the presence of time delays be accounted for in the control design process. In [6, 7] the quadratic stabilization technique was extended to design robust memoryless (delay-independent) feedback controllers for systems with uncertain time delays. The results of [6, 7] were further extended in [8–10] to design robust controllers for systems with uncertain parameter variations and time delays. However, the framework of [8–10] is based on a fixed quadratic Lyapunov function which, as mentioned above, can be conservative for systems with constant real parameter uncertainty.

In this paper we unify the memoryless stabilization technique of [6, 7] for systems with time delays and the parameter-dependent Lyapunov function approach of [3–5] for systems with constant real parameter uncertainty to develop robust full-state feedback controllers guaranteeing robust stability in the face of parametric system uncertainty and system delay.

2. Robust stabilization of systems with parametric uncertainty and time delay

In this section we introduce the robust stabilization problem for systems with constant parameter uncertainty and system state delay. This problem involves a set

$\mathscr{U} \subset \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ of uncertain perturbations $(\Delta A, \Delta A_{\rm d})$ of nominal system matrices A and $A_{\rm d}$, respectively. Specifically, given the nth-order dynamical system

$$\dot{x}(t) = (A + \Delta A)x(t) + (A_{\rm d} + \Delta A_{\rm d})x(t - \tau_{\rm d}) + Bu(t), \quad t \in [0, \infty), \quad \tau_{\rm d} > 0,$$

$$x(t) = \phi(t), \quad t \in [-\tau_{\rm d}, 0], \quad x(0) = \phi(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $\phi : [-\tau_{\rm d}, 0] \to \mathbb{R}^n$ is a continuous vector valued function specifying the initial state of the system, determine a full-state feedback controller

$$u(t) = Kx(t), \tag{2}$$

such that the closed-loop system (1), (2) is asymptotically stable for all $(\Delta A, \Delta A_{\rm d}) \in \mathscr{U}$ and $\tau_{\rm d} > 0$. The uncertainty set $\mathscr{U}$ is defined by

$$\mathscr{U} \triangleq \{(\Delta A, \Delta A_{\rm d}) : \Delta A = B_0 F C_0, \Delta A_{\rm d} = B_{\rm d} F C_{\rm d}, F \in \mathscr{F}\}, \tag{3}$$

where $\mathscr{F}$ satisfies

$$\mathscr{F} \triangleq \{F \in \mathbb{S}^{m_0} : M_1 \leqslant F \leqslant M_2\},$$

and where $B_0, B_{\rm d} \in \mathbb{R}^{n\times m_0}$, $C_0, C_{\rm d} \in \mathbb{R}^{m_0\times n}$, are fixed matrices denoting the structure of the uncertainty, $F \in \mathbb{S}^{m_0\times m_0}$ is an uncertain symmetric matrix, $M_1, M_2 \in \mathbb{S}^{m_0\times m_0}$ are given symmetric matrices such that $M \triangleq M_2 - M_1 \in \mathbb{P}^{m_0\times m_0}$. Thus, for each uncertain variation $(\Delta A, \Delta A_{\rm d}) \in \mathscr{U}$, the closed-loop system (1), (2) can be written as

$$\dot{x}(t) = (A + BK + \Delta A)x(t) + (A_{\rm d} + \Delta A_{\rm d})x(t - \tau_{\rm d}), \quad t \in [0, \infty), \quad \tau_{\rm d} > 0, \tag{4}$$

or, equivalently,

$$\dot{x}(t) = (\tilde{A} + B_0\hat{F}C_0)x(t) + (A_{\rm d} + B_{\rm d}FC_{\rm d})x(t - \tau_{\rm d}), \quad t \in [0, \infty), \quad \tau_{\rm d} > 0, \tag{5}$$

where $\hat{F} \triangleq F - M_1$, $\tilde{A} = \hat{A} + BK$, and $\hat{A} \triangleq A + B_0 M_1 C_0$.

Next, define the set $\mathscr{N}$ of compatible scaling matrices N by

$$\mathscr{N} \triangleq \{N \in \mathbb{R}^{m\times m} : N^{\rm T}(F - M_1) = (F - M_1)N \geqslant 0, F \in \mathscr{F}\}. \tag{6}$$

As noted in [3, 5, 11] the condition that $N^{\rm T}(F - M_1) = (F - M_1)N$, $F \in \mathscr{F}$, allows for nondiagonal real uncertain blocks F while accounting for structure in the uncertainty.

The next result provides a sufficient condition for the robust stability of Eq. (1) for all $(\Delta A, \Delta A_{\rm d}) \in \mathscr{U}$ and $\tau_{\rm d} > 0$. For the statement of next result let $\alpha > 0$ be a given scalar and define the notation

$$R_0 \triangleq [M^{-1} - NC_0B_0 - 2\alpha^{-2}NC_0(A_{\rm d}A_{\rm d}^{\rm T} + B_{\rm d}\hat{R}_1B_{\rm d}^{\rm T})C_0^{\rm T}N^{\rm T}]$$
$$+ [M^{-1} - NC_0B_0 - 2\alpha^{-2}NC_0(A_{\rm d}A_{\rm d}^{\rm T} + B_{\rm d}\hat{R}_1B_{\rm d}^{\rm T})C_0^{\rm T}N^{\rm T}]^{\rm T},$$

where $\hat{R}_1$ is an $n\times n$ nonnegative-definite matrix.

Theorem 2.1. *Let K be given, let $\alpha > 0$, and let $\hat{R}_1, \hat{R}_2 \in \mathbb{N}^m$ be such that $FC_\mathrm{d}C_\mathrm{d}^\mathrm{T}F \leqslant \hat{R}_1$ and $F^2 \leqslant \hat{R}_2$ for all $F \in \mathscr{F}$. Assume $N \in \mathscr{N}$ and $R_0 > 0$. Furthermore, assume there exists an $n \times n$ positive definite matrix P such that*

$$0 = \tilde{A}^\mathrm{T}P + P\tilde{A} + \alpha^2 I_n + 2\alpha^{-2}P(A_\mathrm{d}A_\mathrm{d}^\mathrm{T} + A_\mathrm{d}C_\mathrm{d}^\mathrm{T}C_\mathrm{d}A_\mathrm{d}^\mathrm{T} + B_\mathrm{d}\hat{R}_1B_\mathrm{d}^\mathrm{T} + B_\mathrm{d}\hat{R}_2B_\mathrm{d}^\mathrm{T})P$$
$$+ (C_0 + NC_0\tilde{A} + B_0^\mathrm{T}P)^\mathrm{T}R_0^{-1}(C_0 + NC_0\tilde{A} + B_0^\mathrm{T}P) + R, \tag{7}$$

where R is an $n \times n$ positive definite matrix. Then, the function

$$V(x) = x^\mathrm{T}[P + C_0^\mathrm{T}(F - M_1)NC_0]x + \alpha^2 \int_{t-\tau_\mathrm{d}}^{t} x^\mathrm{T}(s)x(s)\,\mathrm{d}s, \tag{8}$$

is a Lyapunov function that guarantees that the uncertain closed-loop system (1), (2) (or, equivalently Eq. (4)) is asymptotically stable for all $(\Delta A, \Delta A_\mathrm{d}) \in \mathscr{U}$ and $\tau_\mathrm{d} > 0$.

Proof. In order to prove the asymptotic stability of Eq. (5) for all $(\Delta A, \Delta A_\mathrm{d}) \in \mathscr{U}$, consider the Lyapunov function candidate given by Eq. (8). Since P is positive definite and $N \in \mathscr{N}$, it follows that $V(x)$ defined by Eq. (8) is positive definite for all nonzero x. The corresponding Lyapunov derivative along the trajectories $x(t)$, $t \geqslant 0$, of the closed-loop system (4) is given by

$$\dot{V}(x(t)) = \dot{x}^\mathrm{T}(t)[P + C_0^\mathrm{T}\hat{F}NC_0]x(t) + x^\mathrm{T}(t)[P + C_0^\mathrm{T}\hat{F}NC_0]\dot{x}(t)$$
$$+ \alpha^2 \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t-\tau_\mathrm{d}}^{t} x^\mathrm{T}(s)x(s)\,\mathrm{d}s\right], \quad t \geqslant 0, \tag{9}$$

or, using Eq. (5) and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t-\tau_\mathrm{d}}^{t} x^\mathrm{T}(s)x(s)\,\mathrm{d}s\right] = x^\mathrm{T}(t)x(t) - x^\mathrm{T}(t-\tau_\mathrm{d})x(t-\tau_\mathrm{d}),$$

Eq. (9) becomes

$$\dot{V}(x(t)) = x^\mathrm{T}(t)[[\tilde{A}^\mathrm{T}P + P\tilde{A} + \alpha^2 I_n] + C_0^\mathrm{T}\hat{F}[B_0^\mathrm{T}P + NC_0\tilde{A}] + [B_0^\mathrm{T}P + NC_0\tilde{A}]^\mathrm{T}$$
$$\times \hat{F}C_0 + C_0^\mathrm{T}\hat{F}[NC_0B_0 + (NC_0B_0)^\mathrm{T}]\hat{F}C_0]x(t) + 2x^\mathrm{T}(t-\tau_\mathrm{d})$$
$$\times (A_\mathrm{d} + B_\mathrm{d}FC_\mathrm{d})^\mathrm{T}(P + C_0^\mathrm{T}N^\mathrm{T}\hat{F}C_0)x(t) - \alpha^2 x^\mathrm{T}(t-\tau_\mathrm{d})x(t-\tau_\mathrm{d}), \quad t \geqslant 0. \tag{10}$$

Next, noting that $M_1 \leqslant F \leqslant M_2$ for all $F \in \mathscr{F}$ is equivalent to $\hat{F}M^{-1}\hat{F} \leqslant \hat{F}$, for all $F \in \mathscr{F}$ [11], adding and subtracting

$$2x^\mathrm{T}(t)C_0^\mathrm{T}[\hat{F}M^{-1}\hat{F} - \hat{F}]C_0x(t),$$

$$\alpha^{-2}x^\mathrm{T}(t)(P + C_0^\mathrm{T}N^\mathrm{T}\hat{F}C_0)(A_\mathrm{d} + B_\mathrm{d}FC_\mathrm{d})(A_\mathrm{d} + B_\mathrm{d}FC_\mathrm{d})^\mathrm{T}(P + C_0^\mathrm{T}\hat{F}NC_0)x(t),$$

to and from Eq. (10), respectively, and grouping terms yield

$$\begin{aligned}\dot{V}(x(t)) = x^{\mathrm{T}}(t)[[\tilde{A}^{\mathrm{T}}P + P\tilde{A} + \alpha^2 I_n] + C_0^{\mathrm{T}}\hat{F}(C_0 + NC_0\tilde{A} + B_0^{\mathrm{T}}P)\\ + (C_0 + NC_0\tilde{A} + B_0^{\mathrm{T}}P)^{\mathrm{T}}\hat{F}C_0 - C_0^{\mathrm{T}}\hat{F}[(M^{-1} - NC_0B_0)\\ + (M^{-1} - NC_0B_0)^{\mathrm{T}}]\hat{F}C_0]x(t) + 2x^{\mathrm{T}}(t)C_0^{\mathrm{T}}[\hat{F}M^{-1}\hat{F} - \hat{F}]C_0x(t)\\ - p^{\mathrm{T}}p + \alpha^{-2}x^{\mathrm{T}}(t)(P + C_0^{\mathrm{T}}N^{\mathrm{T}}\hat{F}C_0)(A_{\mathrm{d}} + B_{\mathrm{d}}FC_{\mathrm{d}})\\ \times (A_{\mathrm{d}} + B_{\mathrm{d}}FC_{\mathrm{d}})^{\mathrm{T}}(P + C_0^{\mathrm{T}}\hat{F}NC_0)x(t), \quad t \geqslant 0,\end{aligned} \tag{11}$$

where $p \triangleq [\alpha^{-1}(A_{\mathrm{d}} + B_{\mathrm{d}}FC_{\mathrm{d}})^{\mathrm{T}}(P + C_0^{\mathrm{T}}\hat{F}NC_0)x(t) - \alpha x(t - \tau_{\mathrm{d}})]$. Now expanding the last term in Eq. (11), adding and subtracting

$$\alpha^{-2}x^{\mathrm{T}}(t)P(A_{\mathrm{d}} + B_{\mathrm{d}}FC_{\mathrm{d}})(A_{\mathrm{d}} + B_{\mathrm{d}}FC_{\mathrm{d}})^{\mathrm{T}}Px(t),$$

$$\alpha^{-2}x^{\mathrm{T}}(t)C_0^{\mathrm{T}}N^{\mathrm{T}}\hat{F}C_0(A_{\mathrm{d}} + B_{\mathrm{d}}FC_{\mathrm{d}})(A_{\mathrm{d}} + B_{\mathrm{d}}FC_{\mathrm{d}})^{\mathrm{T}}C_0^{\mathrm{T}}\hat{F}NC_0x(t),$$

to and from Eq. (11), and grouping terms yield

$$\begin{aligned}\dot{V}(x(t)) = x^{\mathrm{T}}(t)[[\tilde{A}^{\mathrm{T}}P + P\tilde{A} + \alpha^2 I_n] + C_0^{\mathrm{T}}\hat{F}(C_0 + NC_0\tilde{A} + B_0^{\mathrm{T}}P)\\ + (C_0 + NC_0\tilde{A} + B_0^{\mathrm{T}}P)^{\mathrm{T}}\hat{F}C_0 - C_0^{\mathrm{T}}\hat{F}[(M^{-1} - NC_0B_0)\\ + (M^{-1} - NC_0B_0)^{\mathrm{T}}]\hat{F}C_0]x(t) + 2x^{\mathrm{T}}(t)C_0^{\mathrm{T}}[\hat{F}M^{-1}\hat{F} - \hat{F}]C_0x(t)\\ - p^{\mathrm{T}}p - q^{\mathrm{T}}q + 2\alpha^{-2}x^{\mathrm{T}}(t)P[A_{\mathrm{d}}A_{\mathrm{d}}^{\mathrm{T}} + B_{\mathrm{d}}FC_{\mathrm{d}}A_{\mathrm{d}}^{\mathrm{T}} + A_{\mathrm{d}}C_{\mathrm{d}}^{\mathrm{T}}FB_{\mathrm{d}}^{\mathrm{T}}\\ + B_{\mathrm{d}}FC_{\mathrm{d}}C_{\mathrm{d}}^{\mathrm{T}}FB_{\mathrm{d}}^{\mathrm{T}}]Px(t) + 2\alpha^{-2}x^{\mathrm{T}}(t)C_0^{\mathrm{T}}N^{\mathrm{T}}\hat{F}C_0[A_{\mathrm{d}}A_{\mathrm{d}}^{\mathrm{T}} + B_{\mathrm{d}}FC_{\mathrm{d}}A_{\mathrm{d}}^{\mathrm{T}}\\ + A_{\mathrm{d}}C_{\mathrm{d}}^{\mathrm{T}}FB_{\mathrm{d}}^{\mathrm{T}} + B_{\mathrm{d}}FC_{\mathrm{d}}C_{\mathrm{d}}^{\mathrm{T}}FB_{\mathrm{d}}^{\mathrm{T}}]C_0^{\mathrm{T}}\hat{F}NC_0x(t), \quad t \geqslant 0,\end{aligned} \tag{12}$$

where $q \triangleq \alpha^{-1}(A_{\mathrm{d}} + B_{\mathrm{d}}FC_{\mathrm{d}})^{\mathrm{T}}(P - C_0^{\mathrm{T}}\hat{F}NC_0)x(t)$. Next, adding and subtracting

$$2\alpha^{-2}x(t)P[A_{\mathrm{d}}C_{\mathrm{d}}^{\mathrm{T}}C_{\mathrm{d}}A_{\mathrm{d}}^{\mathrm{T}} + B_{\mathrm{d}}F^2B_{\mathrm{d}}^{\mathrm{T}}]Px(t),$$

$$2\alpha^{-2}x^{\mathrm{T}}(t)C_0^{\mathrm{T}}N^{\mathrm{T}}\hat{F}C_0[A_{\mathrm{d}}A_{\mathrm{d}}^{\mathrm{T}} + B_{\mathrm{d}}FC_{\mathrm{d}}C_{\mathrm{d}}^{\mathrm{T}}FB_{\mathrm{d}}^{\mathrm{T}}]C_0^{\mathrm{T}}\hat{F}NC_0x(t),$$

to and from Eq. (12) and using the fact that $FC_{\mathrm{d}}C_{\mathrm{d}}^{\mathrm{T}}F \leqslant \hat{R}_1$ and $F^2 \leqslant \hat{R}_2$ for all $F \in \mathscr{F}$ it follows that

$$\begin{aligned}\dot{V}(x(t)) = x^{\mathrm{T}}(t)[[\tilde{A}^{\mathrm{T}}P + P\tilde{A} + \alpha^2 I_n + 2\alpha^{-2}P(A_{\mathrm{d}}A_{\mathrm{d}}^{\mathrm{T}} + A_{\mathrm{d}}C_{\mathrm{d}}^{\mathrm{T}}C_{\mathrm{d}}A_{\mathrm{d}}^{\mathrm{T}}\\ + B_{\mathrm{d}}\hat{R}_1B_{\mathrm{d}}^{\mathrm{T}} + B_{\mathrm{d}}\hat{R}_2B_{\mathrm{d}}^{\mathrm{T}}] + C_0^{\mathrm{T}}\hat{F}(C_0 + NC_0\tilde{A} + B_0^{\mathrm{T}}P)\\ + (C_0 + NC_0\tilde{A} + B_0^{\mathrm{T}}P)^{\mathrm{T}}\hat{F}C_0 - C_0^{\mathrm{T}}\hat{F}R_0\hat{F}C_0]x(t)\\ + 2x^{\mathrm{T}}(t)C_0^{\mathrm{T}}[\hat{F}M^{-1}\hat{F} - \hat{F}]C_0x(t) - p^{\mathrm{T}}p - q^{\mathrm{T}}q - 2r^{\mathrm{T}}r - 2s^{\mathrm{T}}s, \quad t \geqslant 0,\end{aligned} \tag{13}$$

where $r \triangleq \alpha^{-1}[C_d A_d^T P - F B_d^T P]x(t)$ and $s \triangleq (A_d - B_d F C_d)^T C_0^T N^T \hat{F} C_0 x(t)$. Now, using Eq. (7)

$$\dot{V}(x(t)) = -x^T(t)[R + z^T z]x(t) - p^T p - q^T q + 2x^T(t) C_0^T [\hat{F} M^{-1} \hat{F} - \hat{F}] C_0 x(t), \quad t \geqslant 0, \tag{14}$$

where $z \triangleq R_0^{-1/2}(C_0 + N C_0 \tilde{A} + B_0^T P) - R_0^{1/2} \hat{F} C_0$. Since R is positive definite and $F \in \mathscr{F}$ it follows that $\dot{V}(x(t)) < 0$, $x(t) \neq 0$, $t \geqslant 0$, and hence the uncertain closed-loop system (5) is asymptotically stable for all $(\Delta A, \Delta A_d) \in \mathscr{U}$ and $\tau_d > 0$. □

Remark 2.1. Note that the Lyapunov function $V(x)$ given by Eq. (8) is explicitly dependent on the uncertain parameters F. In the terminology of [3, 5] this is a parameter-dependent Lyapunov function. Since the parameter-dependent Lyapunov function $V(x)$ explicitly depends on the uncertain parameters F, its ability to guarantee robust stability with respect to time-varying parameter variations is curtailed, thus reducing conservatism with respect to constant real parameter uncertainty. Specifically, if F were permitted to be time-varying then the terms involving $\dot{F}(t)$, $t \geqslant 0$, would potentially subvert the negative definiteness of $\dot{V}(x)$.

3. Sufficient conditions for robust stabilization of uncertain systems with time delay

In this section we present the main theorem characterizing full-state feedback controllers for systems with constant real parameter uncertainty and time delay given by Eq. (1). Specifically, using a constructive procedure, a state feedback gain K is obtained that solves the robust stabilization problem for uncertain time delay systems. For the statement of this result let R_1 and R_2 be $n \times n$ and $m \times m$ positive definite matrices, respectively, and for notational convenience define

$$\tilde{C} \triangleq C_0 + N C_0 \hat{A}, \quad A_P \triangleq \hat{A} + B_0 R_0^{-1} \tilde{C},$$
$$P_a \triangleq B^T P + B^T C_0^T N^T R_0^{-1}(B_0^T P + \tilde{C}), \quad R_{2a} \triangleq R_2 + B^T C_0^T N^T R_0^{-1} N C_0 B,$$

for arbitrary $P \in \mathbb{R}^{n \times n}$.

Theorem 3.1. *Let* $\alpha > 0$. *Assume* $N \in \mathscr{N}$, $R_0 > 0$, *and let* $\hat{R}_1, \hat{R}_2 \in \mathbb{N}^m$ *be such that* $F C_d C_d^T F \leqslant \hat{R}_1$ and $F^2 \leqslant \hat{R}_2$ for all $F \in \mathscr{F}$. *Furthermore, suppose there exists an* $n \times n$ *positive definite matrix* P *satisfying*

$$0 = A_P^T P + P A_P + R_1 + \alpha^2 I_n + \tilde{C}^T R_0^{-1} \tilde{C} + [2\alpha^{-2} P(A_d A_d^T + A_d C_d^T C_d A_d^T + B_d \hat{R}_1 B_d^T + B_d \hat{R}_2 B_d^T) P + P B_0 R_0^{-1} B_0^T P] - P_a^T R_{2a}^{-1} P_a, \tag{15}$$

and let K be given by

$$K = -R_{2a}^{-1} P_a. \tag{16}$$

Then the closed-loop system (1), (2) is asymptotically stable for all $(\Delta A, \Delta A_d) \in \mathscr{U}$ and $\tau_d > 0$.

Proof. With K given by Eq. (16), it follows that Eq. (15) is equivalent to

$$0 = \tilde{A}^{\mathrm{T}}P + P\tilde{A} + \alpha^2 I_n + 2\alpha^{-2}P(A_{\mathrm{d}}A_{\mathrm{d}}^{\mathrm{T}} + A_{\mathrm{d}}C_{\mathrm{d}}^{\mathrm{T}}C_{\mathrm{d}}A_{\mathrm{d}}^{\mathrm{T}} + B_{\mathrm{d}}\hat{R}_1B_{\mathrm{d}}^{\mathrm{T}} + B_{\mathrm{d}}\hat{R}_2B_{\mathrm{d}}^{\mathrm{T}})P$$
$$+ [C_0 + NC_0\tilde{A} + B_0^{\mathrm{T}}P]^{\mathrm{T}}R_0^{-1}[C_0 + NC_0\tilde{A} + B_0^{\mathrm{T}}P] + R_1 + K^{\mathrm{T}}R_2K.$$

It now follows from Theorem 2.1 that the closed-loop system (1), (2) is asymptotically stable for all $(\Delta A, \Delta A_{\mathrm{d}}) \in \mathcal{U}$ and $\tau_{\mathrm{d}} > 0$. □

Remark 3.1. Note that setting $N = 0$ and $-M_1 = M_2 = \gamma^{-1}I$, where $\gamma > 0$, Theorem 3.1 specializes to Theorem 1 of [8] for constant delay and time-varying uncertainty.

Remark 3.2. Theorem 3.1 presents sufficient conditions for designing robust full-state feedback controllers for time delay systems with constant real parameter uncertainty. Using the fixed-structure controller synthesis framework developed in [5] these results can be readily extended to fixed-order (i.e., full- and reduced-order) dynamic compensation.

4. Illustrative numerical example

In this section we present an illustrative numerical example to demonstrate the proposed robust stabilization approach for uncertain time delay systems. The design equation (15) was solved using a homotopy continuation algorithm. Specifically, we parameterize Eq (15) as

$$0 = A_P^{\mathrm{T}}P_{i+1} + P_{i+1}A_P + R_1 + \alpha^2 I_n + \tilde{C}^{\mathrm{T}}R_0^{-1}\tilde{C} + \lambda[2\alpha^{-2}P_i(A_{\mathrm{d}}A_{\mathrm{d}}^{\mathrm{T}} + A_{\mathrm{d}}C_{\mathrm{d}}^{\mathrm{T}}C_{\mathrm{d}}A_{\mathrm{d}}^{\mathrm{T}}$$
$$+ B_{\mathrm{d}}\hat{R}_1B_{\mathrm{d}}^{\mathrm{T}} + B_{\mathrm{d}}\hat{R}_2B_{\mathrm{d}}^{\mathrm{T}})P_i + P_iB_0R_0^{-1}B_0^{\mathrm{T}}P_i] - P_{a_{i+1}}^{\mathrm{T}}R_{2a}^{-1}P_{a_{i+1}}, \quad (17)$$

where

$$P_{a_{i+1}} \triangleq B^{\mathrm{T}}P_{i+1} + B^{\mathrm{T}}C_0^{\mathrm{T}}N^{\mathrm{T}}R_0^{-1}(B_0^{\mathrm{T}}P_{i+1} + \tilde{C}),$$

and $\lambda \in [0, 1]$. The algorithm is initialized with $\lambda = 0$ and solves for P_{i+1} with small increments in λ until $\lambda = 1$. Next, with $\lambda = 1$, we iteratively solve for P_{i+1}. Alternatively, by identifying the terms within the square brackets as H_∞-type terms, Eq. (15) can also be solved as a modified H_∞ Riccati equation with a cross weighting term [12].

Example 4.1. Consider the uncertain delay dynamical system (1) with problem data

$$A = \begin{bmatrix} 0.1 & 1 \\ 1 & -1.5 \end{bmatrix}, \quad A_{\mathrm{d}} = \begin{bmatrix} -0.25 & -0.13 \\ -0.475 & -0.25 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Note that the nominal plant dynamics A is unstable. We consider $\pm 20\%$ uncertainty in the (2, 2) element of the system dynamics matrix A and no uncertainty in A_{d}. Using the uncertainty structure given by Eq. (3), the uncertain system matrices are given by

$\Delta A = B_0 f C_0$ and $\Delta A_d = B_d f C_d$, where $B_0^T = C_0 = [0\ \ 1]$, $B_d^T = C_d = [0\ \ 0]$, and $-0.3 \leqslant f \leqslant 0.3$. In this case, $M_1 = -0.3$ and $M_2 = 0.3$. Next, choosing the design variables $R_1 = I_2$, $R_2 = 1$, $\alpha = 2$, and $N = 1$ and using the homotopy continuation algorithm outlined above a positive definite solution to Eq. (15) is given by

$$P = \begin{bmatrix} 4.9746 & 2.3264 \\ 2.3264 & 2.4991 \end{bmatrix},$$

and

$$K = [-4.9746 \quad -2.3264].$$

5. Conclusion

The parameter-dependent Lyapunov function framework used to address the problem of robust stabilization for constant real parameter uncertainty [3–5] was merged with the delay stabilization framework [6, 7] to develop full-state feedback controllers guaranteeing robust stability in the face of parameter uncertainty and time delay.

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