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# Robust nonlinear feedback control for uncertain linear systems with nonquadratic performance criteria<sup>1</sup>

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## Abstract

In this paper we develop a unified framework to address the problem of optimal nonlinear robust control for linear uncertain systems. Specifically, we transform a given robust control problem into an optimal control problem by properly modifying the cost functional to account for the system uncertainty. As a consequence, the resulting solution to the modified optimal control problem guarantees robust stability and performance for a class of nonlinear uncertain systems. The overall framework generalizes the classical Hamilton–Jacobi–Bellman conditions to address the design of robust nonlinear optimal controllers for uncertain linear systems. © 1998 Elsevier Science B.V. All rights reserved.

*Keywords:* Robust control; Nonlinear control; Nonquadratic cost; Hamilton–Jacobi–Bellman conditions; Lyapunov bounding functions

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## 1. Introduction

Unavoidable discrepancies between system models and real-world systems can result in degradation of control-system performance including instability [8, 24]. Ideally, feedback control systems should be designed to be robust with respect to system uncertainties. In designing robust controllers there are two principal issues, namely, stability robustness and performance robustness. Stability robustness addresses the problem of guaranteeing stability of the closed-loop system for plant perturbations within a specified class of uncertainties. In addition to guaranteeing robust stability, it is often desirable to minimize the worst-case performance degradation within a given robust stability range.

One approach to robust control design involves modeling system uncertainty by means of the  $H_\infty$  norm and then using  $H_\infty$  theory to guarantee robust stability and robust performance. In this case it is well known that nonlinear controllers offer no advantage over linear controllers [1, 15, 19]. However, these results are restricted to unstructured uncertainty [10, 15, 19],  $H_\infty$  performance [16], and quadratic stability [17]. In the case of structured parametric uncertainty with nonquadratic performance criteria nonlinear controllers can yield better robust performance than linear controllers. In fact it is not unreasonable to conjecture that the best controller that solves the robust, linear-quadratic problem with structured parametric uncertainty is a nonlinear controller.

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In a recent paper by Bernstein [3] the current status of nominal (completely accurate) nonlinear-nonquadratic problems was presented in a simplified and tutorial manner. The basic underlying ideas of the results in [3] are based on the fact that the steady-state solution of the Hamilton–Jacobi–Bellman equation is a Lyapunov function for the nominal nonlinear system thus guaranteeing both optimality and stability [14]. In this paper we extend the framework developed in [3] to address the problem of optimal nonlinear robust control. Specifically, we transform a robust nonlinear control problem into an optimal control problem. This is accomplished by properly modifying the cost functional to account for system uncertainty so that the solution of the modified optimal nonlinear control problem serves as the solution to the robust control problem. The present framework generalizes the linear guaranteed cost control approach for addressing robust stability and performance [4, 6, 7, 18, 20] to linear uncertain systems controlled via nonlinear controllers.

The main contribution of this paper is a methodology for designing nonlinear controllers which provide both robust stability and robust performance over a prescribed range of system uncertainty. The present framework extends the guaranteed cost linear control approach for linear uncertain systems [6, 7] to nonlinear control by utilizing a performance bound to provide robust performance in addition to robust stability. In particular, the performance bound can be evaluated in closed form as long as the nonlinear-nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees robust stability over a prescribed uncertainty set. This Lyapunov function is shown to be the solution to the steady-state form of the Hamilton–Jacobi–Bellman equation for the nominal system and plays a key role in constructing the optimal nonlinear robust control law. Hence the overall framework provides for a generalization of the Hamilton–Jacobi–Bellman conditions for addressing the design of robust nonlinear optimal controllers for linear uncertain systems.

A key feature of the present framework is that since the necessary Hamilton–Jacobi–Bellman optimality conditions are obtained for a nonlinear-nonquadratic performance functional, *globally* optimal controllers are guaranteed to provide both robust stability and performance. Of course, since our approach allows us to construct globally optimal controllers that minimize a given Hamiltonian, the resulting robust nonlinear controllers provide the best worst-case performance over the robust stability range.

We emphasize that our controllers are predicated on an *inverse optimal robust control problem* [9]. In particular, to avoid the complexity in solving the steady-state robustified Hamilton–Jacobi–Bellman equation we do not attempt to minimize a *given* cost functional over a prescribed range of system parametric uncertainty, but rather, we parameterize a family of robustly stabilizing controllers that minimize some *derived* cost functional that provides flexibility in specifying the robust control law. The performance integrand is shown to explicitly depend on the solution to a set of unidirectionally coupled Riccati/Lyapunov equations that characterize the robustly stabilizing controller. Hence, by varying the free parameters in the Riccati/Lyapunov equations the proposed framework can be used to characterize a class of globally robustly stabilizing controllers that can meet closed-loop system response requirements over a prescribed range of system parametric uncertainty. Finally, the results presented in this paper can be viewed as extending the nonlinear disturbance rejection control problem using Hamilton–Jacobi–Isaacs equations [12, 13, 21, 22] to nonlinear robust control for systems with parametric uncertainty.

In this paper we use the following standard notation. Let  $\mathbb{R}$  denote real numbers and let  $\mathbb{R}^{n \times m}$  denote real  $n \times m$  matrices. Let  $\mathbb{N}^{n \times n}$  ( $\mathbb{P}^{n \times n}$ ) denote  $n \times n$  nonnegative (positive) definite matrices. Furthermore,  $A \geq 0$  ( $A > 0$ ) denotes the fact that the Hermitian matrix  $A$  is nonnegative (positive) definite and  $A \geq B$  ( $A > B$ ) denotes the fact that  $A - B \geq 0$  ( $A - B > 0$ ).

## 2. Robust optimal control for nonlinear uncertain systems

In this section we consider a control problem involving a notion of optimality with respect to an *auxiliary cost* which guarantees a bound on the worst-case value of a nonlinear-nonquadratic cost criterion over a prescribed uncertainty set. The optimal robust feedback controllers provide a transparent generalization of the Hamilton–Jacobi–Bellman conditions for time-invariant, infinite horizon problems for addressing robust nonlinear feedback controllers for nonlinear systems. To address the robust optimal control problem let  $\mathcal{D} \subset \mathbb{R}^n$

be an open set and let  $\mathcal{C} \subset \mathbb{R}^m$ , where  $0 \in \mathcal{D}$  and  $0 \in \mathcal{C}$ . Furthermore, let  $\tilde{\mathcal{F}} \subset \{\tilde{f}: \mathcal{D} \times \mathcal{C} \rightarrow \mathbb{R}^n: \tilde{f}(0,0)=0\}$  denote the class of uncertain closed-loop nonlinear systems with  $\tilde{f}_0(\cdot, \cdot) \in \tilde{\mathcal{F}}$  defining the nominal nonlinear system. Next, consider the controlled uncertain system

$$\dot{x}(t) = \tilde{f}(x(t), u(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{1}$$

where  $\tilde{f}(\cdot, \cdot) \in \tilde{\mathcal{F}}$  and the control  $u(\cdot)$  is restricted to the class of *admissible* controls consisting of measurable functions  $u(\cdot)$  such that  $u(t) \in \mathcal{U}$  for all  $t \geq 0$  where the control constraint set  $\mathcal{U} \subset \mathcal{C}$  is given. We assume  $0 \in \mathcal{U}$ . A measurable mapping  $\phi: \mathcal{D} \rightarrow \mathcal{U}$  satisfying  $\phi(0) = 0$  is called a *control law*. If  $u(t) = \phi(x(t))$ , where  $\phi(\cdot)$  is a control law and  $x(t)$  satisfies (1), then  $u(\cdot)$  is called a *feedback control*. Given a control law  $\phi(\cdot)$  and a feedback control  $u(t) = \phi(x(t))$ , the *closed-loop system* has the form

$$\dot{x}(t) = \tilde{f}(x(t), \phi(x(t))), \quad x(0) = x_0, \quad t \geq 0, \tag{2}$$

for all  $\tilde{f}(\cdot, \cdot) \in \tilde{\mathcal{F}}$ . We assume that the mapping  $\phi: \mathcal{D} \rightarrow \mathcal{U}$  satisfies sufficient regularity conditions such that Eq. (2) has a unique solution forward in time. Specifically, we assume that  $\tilde{f}(\cdot, \cdot) \in \tilde{\mathcal{F}}$  is smooth ( $C^\infty$  mapping) and Lipschitz defined in a neighborhood of the origin in  $\mathcal{D} \times \mathcal{C}$ .

Next we present an extension of Theorem 4.1 of [3] for characterizing robust feedback controllers that guarantee robust stability over a class of nonlinear uncertain systems and minimize an auxiliary performance functional. For the statement of this result let  $\tilde{L}: \mathcal{D} \times \mathcal{U} \rightarrow \mathbb{R}$  and define the set of asymptotically stabilizing controllers for the nominal nonlinear system  $\tilde{f}_0(\cdot, \cdot)$  by

$$\mathcal{S}(x_0) \triangleq \{u(\cdot): u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by Eq. (1) satisfies } x(t) \rightarrow 0 \text{ as } t \rightarrow \infty \text{ with } \tilde{f}(\cdot, \cdot) = \tilde{f}_0(\cdot, \cdot)\}.$$

**Theorem 2.1.** *Consider the controlled system (1) with performance functional*

$$\tilde{J}_{\tilde{f}}(x_0, u(\cdot)) \triangleq \int_0^\infty \tilde{L}(x(t), u(t)) dt, \tag{3}$$

where  $\tilde{f}(\cdot, \cdot) \in \tilde{\mathcal{F}}$  and  $u(\cdot)$  is an admissible control. Assume there exist functions  $V: \mathcal{D} \rightarrow \mathbb{R}$ ,  $\tilde{\Gamma}: \mathcal{D} \times \mathcal{U} \rightarrow \mathbb{R}$ , and control law  $\phi: \mathcal{D} \rightarrow \mathcal{U}$  where  $V(\cdot)$  is a  $C^1$  function such that

$$V(0) = 0, \tag{4}$$

$$V(x) > 0, \quad x \in \mathcal{D}, \quad x \neq 0, \tag{5}$$

$$\phi(0) = 0, \tag{6}$$

$$V'(x)\tilde{f}(x, \phi(x)) \leq V'(x)\tilde{f}_0(x, \phi(x)) + \tilde{\Gamma}(x, \phi(x)), \quad x \in \mathcal{D}, \quad \tilde{f}(\cdot, \cdot) \in \tilde{\mathcal{F}}, \tag{7}$$

$$V'(x)\tilde{f}_0(x, \phi(x)) + \tilde{\Gamma}(x, \phi(x)) < 0, \quad x \in \mathcal{D}, \quad x \neq 0, \tag{8}$$

$$H(x, \phi(x)) = 0, \quad x \in \mathcal{D}, \tag{9}$$

$$H(x, u) \geq 0, \quad x \in \mathcal{D}, \quad u \in \mathcal{U}, \tag{10}$$

where  $\tilde{f}_0(\cdot, \cdot) \in \tilde{\mathcal{F}}$  defines the nominal system and

$$H(x, u) \triangleq \tilde{L}(x, u) + V'(x)\tilde{f}_0(x, u) + \tilde{\Gamma}(x, u). \tag{11}$$

Then, with the feedback control  $u(\cdot) = \phi(x(\cdot))$ , there exists a neighborhood  $\mathcal{D}_0 \subset \mathcal{D}$  of the origin such that if  $x_0 \in \mathcal{D}_0$ , the solution  $x(t) = 0, t \geq 0$ , of the closed-loop system (2) is locally asymptotically stable for all  $\tilde{f}(\cdot, \cdot) \in \tilde{\mathcal{F}}$ . Furthermore,

$$\sup_{\tilde{f}(\cdot, \cdot) \in \tilde{\mathcal{F}}} \tilde{J}_{\tilde{f}}(x_0, \phi(x(\cdot))) \leq \tilde{J}(x_0, \phi(x(\cdot))) = V(x_0), \tag{12}$$

where

$$\tilde{\mathcal{J}}(x_0, u(\cdot)) \triangleq \int_0^\infty [\tilde{L}(x(t), u(t)) + \tilde{\Gamma}(x(t), u(t))] dt, \quad (13)$$

and where  $u(\cdot)$  is admissible and  $x(t)$ ,  $t \geq 0$ , solves Eq. (1) with  $\tilde{f}(x(t), u(t)) = \tilde{f}_0(x(t), u(t))$ . In addition, if  $x_0 \in \mathcal{D}_0$  then the feedback control  $u(\cdot) = \phi(x(\cdot))$  minimizes  $\tilde{\mathcal{J}}(x_0, u(\cdot))$  in the sense that

$$\tilde{\mathcal{J}}(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} \tilde{\mathcal{J}}(x_0, u(\cdot)). \quad (14)$$

Finally, if  $\mathcal{D} = \mathbb{R}^n$ ,  $\mathcal{U} = \mathbb{R}^m$ , and

$$V(x) \rightarrow \infty \quad \text{as } \|x\| \rightarrow \infty, \quad (15)$$

then the solution  $x(t) = 0$ ,  $t \geq 0$ , of Eq. (2) is globally asymptotically stable for all  $\tilde{f}(\cdot, \cdot) \in \tilde{\mathcal{F}}$ .

**Proof.** Let  $\tilde{f}(\cdot, \cdot) \in \tilde{\mathcal{F}}$ ,  $u(\cdot) = \phi(x(\cdot))$ , and  $x(t)$ ,  $t \geq 0$ , satisfy Eq. (2). Then

$$\dot{V}(x(t)) \triangleq \frac{d}{dt} V(x(t)) = V'(x(t)) \tilde{f}(x(t), \phi(x(t))), \quad t \geq 0. \quad (16)$$

Hence it follows from Eqs. (7) and (8) that

$$\dot{V}(x(t)) < 0, \quad t \geq 0, \quad x(t) \neq 0. \quad (17)$$

Thus, from Eqs. (4), (5), and (17) it follows that  $V(\cdot)$  is a Lyapunov function [26] for Eq. (2), which proves local asymptotic stability of the solution  $x(t) = 0$ ,  $t \geq 0$ , for all  $\tilde{f}(\cdot, \cdot) \in \tilde{\mathcal{F}}$ . Consequently,  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  for all initial conditions  $x_0 \in \mathcal{D}_0$  for some neighborhood of the origin  $\mathcal{D}_0 \subset \mathcal{D}$ . Now Eq. (16) implies that

$$0 = -\dot{V}(x(t)) + V'(x(t)) \tilde{f}(x(t), \phi(x(t))), \quad t \geq 0,$$

and hence, using Eqs. (7) and (9),

$$\begin{aligned} \tilde{L}(x(t), \phi(x(t))) &= -\dot{V}(x(t)) + \tilde{L}(x(t), \phi(x(t))) + V'(x(t)) \tilde{f}(x(t), \phi(x(t))) \\ &\leq -\dot{V}(x(t)) + \tilde{L}(x(t), \phi(x(t))) + V'(x(t)) \tilde{f}_0(x(t), \phi(x(t))) + \tilde{\Gamma}(x(t), \phi(x(t))) \\ &= -\dot{V}(x(t)). \end{aligned}$$

Now, integrating over  $[0, t]$  yields

$$\int_0^t L(x(s), \phi(x(s))) ds \leq -V(x(t)) + V(x_0).$$

Letting  $t \rightarrow \infty$  and noting that  $V(x(t)) \rightarrow 0$  for all  $x_0 \in \mathcal{D}_0$  yields  $\tilde{\mathcal{J}}_f(x_0, \phi(x(\cdot))) \leq V(x_0)$ . Next let  $x(t)$ ,  $t \geq 0$ , satisfy Eq. (2) with  $f(x(t)) = f_0(x(t))$ . Then it follows from Eq. (9) that

$$\begin{aligned} \tilde{L}(x(t), \phi(x(t))) + \tilde{\Gamma}(x(t), \phi(x(t))) &= -\dot{V}(x(t)) + \tilde{L}(x(t), \phi(x(t))) \\ &\quad + V'(x(t)) \tilde{f}_0(x(t), \phi(x(t))) + \tilde{\Gamma}(x(t), \phi(x(t))) \\ &= -\dot{V}(x(t)). \end{aligned}$$

Integrating over  $[0, t]$  yields

$$\int_0^t [\tilde{L}(x(s), \phi(x(s))) + \tilde{\Gamma}(x(s), \phi(x(s)))] ds = -V(x(t)) + V(x_0).$$

Now letting  $t \rightarrow \infty$  yields  $\tilde{\mathcal{J}}(x_0, \phi(x(\cdot))) = V(x_0)$ . Next, let  $u(\cdot) \in \mathcal{S}(x_0)$  and let  $x(\cdot)$  be the solution of Eq. (1) with  $\tilde{f}(\cdot, \cdot) = \tilde{f}_0(\cdot, \cdot)$ . Then it follows that

$$0 = -\dot{V}(x(t)) + V'(x(t)) \tilde{f}_0(x(t), u(t)).$$

Hence

$$\begin{aligned} \tilde{L}(x(t), u(t)) + \tilde{\Gamma}(x(t), u(t)) &= -\dot{V}(x(t)) + \tilde{L}(x(t), u(t)) + V'(x(t))\tilde{f}_0(x(t), u(t)) + \tilde{\Gamma}(x(t), u(t)) \\ &= -\dot{V}(x(t)) + H(x(t), u(t)). \end{aligned}$$

Now using Eqs. (11) and (13) and the fact that  $u(\cdot) \in \mathcal{S}(x_0)$ , it follows that

$$\begin{aligned} \tilde{\mathcal{J}}(x_0, u(\cdot)) &= \int_0^\infty [-\dot{V}(x(t)) + H(x(t), u(t))] dt \\ &= -\lim_{t \rightarrow \infty} V(x(t)) + V(x_0) + \int_0^\infty H(x(t), u(t)) dt \\ &= V(x_0) + \int_0^\infty H(x(t), u(t)) dt \\ &\geq V(x_0) = \tilde{\mathcal{J}}(x_0, \phi(x(\cdot))), \end{aligned}$$

which yields Eq. (14). Finally, global asymptotic stability follows as a direct consequence of the radial unboundedness condition (15).  $\square$

**Remark 2.1.** Note that conditions (4)–(6) assure that  $V(x)$  is a Lyapunov function candidate for the closed-loop system (2). Conditions (7) and (8) imply that  $\dot{V}(x(t)) < 0$  for  $x(\cdot)$  satisfying Eq. (2) for all  $\tilde{f}(\cdot, \cdot) \in \tilde{\mathcal{F}}$  and hence  $V(\cdot)$  is a Lyapunov function guaranteeing robust stability of the closed-loop system (2). It is important to note that condition (8) is a *verifiable* condition since it is independent of the uncertain system parameters  $\tilde{f}(\cdot, \cdot) \in \tilde{\mathcal{F}}$ . To apply Theorem 2.1 we specify a bounding function  $\tilde{\Gamma}(\cdot, \cdot)$  for an uncertainty set  $\tilde{\mathcal{F}}$  such that  $\tilde{\Gamma}(\cdot, \cdot)$  bounds  $\tilde{\mathcal{F}}$  (see Propositions 3.1 and 3.2). Finally, conditions (9) and (10) correspond to the steady-state Hamilton–Jacobi–Bellman conditions for the nominal nonlinear system  $\tilde{f}_0(\cdot, \cdot)$  with the auxiliary cost  $\tilde{\mathcal{J}}(x_0, u(\cdot))$ .

**Remark 2.2.** If  $\tilde{\mathcal{F}}$  consists of only the nominal nonlinear closed-loop system  $\tilde{f}_0(\cdot, \cdot)$ , then  $\tilde{\Gamma}(x, u) = 0$  for all  $x \in \mathcal{D}$  and  $u \in \mathcal{U}$  satisfies Eq. (7) and hence  $\tilde{J}_f(x_0, u(\cdot)) = \tilde{\mathcal{J}}(x_0, u(\cdot))$ . In this case Theorem 2.1 specializes to Theorem 4.1 of [3].

Next, we specialize Theorem 2.1 to linear uncertain systems and provide connections to the quadratic Lyapunov bounding synthesis framework developed in [4, 6]. Specifically, in this case we consider  $\tilde{\mathcal{F}}$  to be the set of uncertain linear systems given by  $\{(A + \Delta A)x + Bu: x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, \Delta A \in \mathcal{A}\}$ , where  $\mathcal{A} \subset \mathbb{R}^{n \times n}$  is a given bounded uncertainty set of the uncertain perturbation  $\Delta A$  of the nominal system  $A$  such that  $0 \in \mathcal{A}$ . In this section no explicit structure is assumed for the elements of  $\mathcal{A}$ . In Section 3 the structure of variations in  $\mathcal{A}$  will be specified. Even though uncertainty in both  $A$  and  $B$  can be considered, for simplicity of exposition we assume  $\Delta B = 0$ . The case  $\Delta B \neq 0$  is treated in [11]. For the following result let  $R_1 \in \mathbb{P}^{n \times n}$  and  $R_2 \in \mathbb{P}^{m \times m}$  be given and define  $S \triangleq BR_2^{-1}B^T$ .

**Corollary 2.1.** Consider the linear uncertain system

$$\dot{x}(t) = (A + \Delta A)x(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{18}$$

with performance functional

$$\tilde{J}_{\Delta A}(x_0, u(\cdot)) \triangleq \int_0^\infty [x^T(t)R_1x(t) + u^T(t)R_2u(t)] dt, \tag{19}$$

where  $u(\cdot)$  is admissible and  $\Delta A \in \mathcal{A}$ . Furthermore, assume there exist  $P \in \mathbb{P}^{n \times n}$  and  $\Omega: \mathbb{P}^{n \times n} \rightarrow \mathbb{N}^{n \times n}$  such that

$$\Delta A^T P + P \Delta A \leq \Omega(P), \quad \Delta A \in \mathcal{A}, \tag{20}$$

and

$$0 = A^T P + PA + R_1 + \Omega(P) - PSP. \quad (21)$$

Then the linear uncertain system given by Eq. (18) is globally asymptotically stable for all  $x_0 \in \mathbb{R}^n$  and  $\Delta A \in \mathcal{A}$ , with the feedback control  $u = \phi(x) \triangleq -R_2^{-1} B^T P x$ , and

$$\sup_{\Delta A \in \mathcal{A}} \tilde{J}_{\Delta A}(x_0, \phi(x(\cdot))) \leq \tilde{\mathcal{J}}(x_0, \phi(x(\cdot))) = x_0^T P x_0, \quad (22)$$

where

$$\tilde{\mathcal{J}}(x_0, u(\cdot)) \triangleq \int_0^\infty [x^T(t)(R_1 + \Omega(P))x(t) + u^T(t)R_2 u(t)] dt, \quad (23)$$

and where  $u(\cdot)$  is admissible and  $x(t)$ ,  $t \geq 0$ , solves Eq. (18) with  $\Delta A = 0$ . Furthermore,

$$\tilde{\mathcal{J}}(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} \tilde{\mathcal{J}}(x_0, u(\cdot)), \quad (24)$$

where  $\mathcal{S}(x_0)$  is the set of asymptotically stabilizing controllers for the nominal system and  $x_0 \in \mathbb{R}^n$ .

**Proof.** The result is a direct consequence of Theorem 2.1 with  $\mathcal{D} = \mathbb{R}^n$ ,  $\mathcal{U} = \mathbb{R}^m$ ,  $\tilde{f}(x, u) = (A + \Delta A)x + Bu$ ,  $\tilde{f}_0(x, u) = Ax + Bu$ ,  $\tilde{L}(x, u) = x^T R_1 x + u^T R_2 u$ ,  $V(x) = x^T P x$ , and  $\tilde{T}(x, u) = x^T \Omega(P)x$ .  $\square$

**Remark 2.3.** The optimal robust feedback control law  $\phi(x)$  in Corollary 2.1 is derived using the properties of  $H(x, u)$  as defined in Theorem 2.1. Specifically, since  $H(x, u) = x^T(A^T P + PA + R_1 + \Omega(P))x + u^T R_2 u + 2x^T P B u$  it follows that  $\partial^2 H / \partial u^2 = R_2 > 0$ . Now,  $\partial H / \partial u = 2R_2 u + 2B^T P x = 0$  gives the unique global minimum of  $H(x, u)$  for all  $\Delta A \in \mathcal{A}$ . Hence, since  $\phi(x)$  minimizes  $H(x, u)$  for all  $\Delta A \in \mathcal{A}$  it follows that  $\phi(x)$  satisfies  $\partial H / \partial u = 0$  or, equivalently,  $\phi(x) = -R_2^{-1} B^T P x$ . Similar remarks hold for the nonlinear robust controllers developed in Section 3.

### 3. Robust nonlinear controllers for uncertain linear systems with polynomial performance criteria

In this section we specialize the results of Section 2 to uncertain linear systems controlled by nonlinear controllers that minimize a polynomial cost functional. Specifically, assume  $\tilde{\mathcal{F}}$  to be the set of uncertain systems given by  $\{(A + \Delta A)x + Bu : x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, \Delta A \in \mathcal{A}\}$ , where  $\mathcal{A} \subset \mathbb{R}^{n \times n}$  is a given bounded uncertainty set of the uncertain perturbation  $\Delta A$  of the nominal system  $A$  such that  $0 \in \mathcal{A}$ . For the following result recall the definition of  $S$  and let  $R_1 \in \mathbb{P}^{n \times n}$ ,  $R_2 \in \mathbb{P}^{m \times m}$ , and  $\hat{R}_k \in \mathbb{N}^{n \times n}$ ,  $k = 2, \dots, r$ , be given where  $r$  is a positive integer.

**Theorem 3.1.** Consider the uncertain system

$$\dot{x}(t) = (A + \Delta A)x(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \quad (25)$$

where  $u(\cdot)$  is admissible and  $\Delta A \in \mathcal{A}$ . Assume there exists  $\Omega : \mathbb{N}^{n \times n} \rightarrow \mathbb{N}^{n \times n}$  such that

$$\Delta A^T P + P \Delta A \leq \Omega(P), \quad \Delta A \in \mathcal{A}, \quad P \in \mathbb{N}^{n \times n}, \quad (26)$$

and there exist  $P \in \mathbb{P}^{n \times n}$ ,  $M_k \in \mathbb{N}^{n \times n}$ ,  $k = 2, \dots, r$ , such that

$$0 = A^T P + PA + R_1 + \Omega(P) - PSP, \quad (27)$$

and

$$0 = (A - SP)^T M_k + M_k (A - SP) + \hat{R}_k + \Omega(M_k), \quad k = 2, \dots, r. \quad (28)$$

Furthermore, let

$$\begin{aligned} \tilde{L}(x, u) = & x^T \left( R_1 + \sum_{k=2}^r (x^T M_k x)^{k-1} \hat{R}_k + \left[ \sum_{k=2}^r (x^T M_k x)^{k-1} M_k \right]^T S \left[ \sum_{k=2}^r (x^T M_k x)^{k-1} M_k \right] \right) x \\ & + u^T R_2 u, \end{aligned} \tag{29}$$

and

$$\tilde{F}(x, u) = x^T \left( \Omega(P) + \sum_{k=2}^r (x^T M_k x)^{k-1} \Omega(M_k) \right) x, \tag{30}$$

where  $u(\cdot)$  is admissible and  $\Delta A \in \mathbf{A}$ . Then the uncertain system (25), with performance functional

$$\tilde{J}_{\Delta A}(x_0, u(\cdot)) \triangleq \int_0^\infty \tilde{L}(x, u) dt, \tag{31}$$

is globally asymptotically stable for all  $x_0 \in \mathbb{R}^n$  and  $\Delta A \in \mathbf{A}$ , with the feedback control  $u = \phi(x) \triangleq -R_2^{-1} B^T (P + \sum_{k=2}^r (x^T M_k x)^{k-1} M_k) x$ . Furthermore,

$$\sup_{\Delta A \in \mathbf{A}} \tilde{J}_{\Delta A}(x_0, \phi(x(\cdot))) \leq \tilde{\mathcal{J}}(x_0, \phi(x(\cdot))) = x_0^T P x_0 + \sum_{k=2}^r \frac{1}{k} (x_0^T M_k x_0)^k, \tag{32}$$

where

$$\tilde{\mathcal{J}}(x_0, u(\cdot)) \triangleq \int_0^\infty [\tilde{L}(x, u) + \tilde{F}(x, u)] dt, \tag{33}$$

and where  $u(\cdot)$  is admissible, and  $x(t)$ ,  $t \geq 0$ , solves Eq. (25) with  $\Delta A = 0$ . In addition,

$$\tilde{\mathcal{J}}(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} \tilde{\mathcal{J}}(x_0, u(\cdot)), \tag{34}$$

where  $\mathcal{S}(x_0)$  is the set of asymptotically stabilizing controllers for the nominal system and  $x_0 \in \mathbb{R}^n$ .

**Proof.** The result is a direct consequence of Theorem 2.1 with  $\mathcal{D} = \mathbb{R}^n$ ,  $\mathcal{U} = \mathbb{R}^m$ ,  $\tilde{f}(x, u) = (A + \Delta A)x + Bu$ ,  $\tilde{f}_0(x, u) = Ax + Bu$ ,  $V(x) = x^T P x + \sum_{k=2}^r (1/k)(x^T M_k x)^k$ ,  $\tilde{L}(x, u)$  given by Eq. (29), and  $\tilde{F}(x, u)$  given by Eq. (30).  $\square$

**Remark 3.1.** Theorem 3.1 generalizes the deterministic version of the stochastic nonlinear–nonquadratic-optimal control problem considered in [25] to the robustness setting. Furthermore, unlike the results of [25], Theorem 3.1 is not limited to sixth-order cost functionals and cubic nonlinear controllers since it addresses a polynomial nonlinear performance criterion.

**Remark 3.2.** Theorem 3.1 requires the solutions of  $r - 1$  modified Riccati equations in Eq. (28) to obtain the optimal robust controller. However, if  $\hat{R}_k = \hat{R}_2$ ,  $k = 3, \dots, r$ , then  $M_k = M_2$ ,  $k = 3, \dots, r$ , satisfies Eq. (28). In this case we require the solution of modified Riccati equation in Eq. (28). This special case is considered in Propositions 3.1 and 3.2.

**Remark 3.3.** As noted in the Introduction the robust controller given in Theorem 3.1 is predicated on an inverse optimal robust control framework. In particular, the robust performance bound (33) is dependent on the design equation (28). However, as shown in Section 4, by varying the free weighting parameters  $R_1, R_2$ ,

and  $\hat{R}_k$ ,  $k=2, \dots, r$ , a class of globally robustly stabilizing controllers can be designed to meet closed-loop system response requirements over a prescribed range of system parametric uncertainty.

**Remark 3.4.** As discussed in Remark 3.3 and [2,3,25] the performance functional (31) is somewhat contrived in the sense that it cannot be arbitrarily specified. However, this performance functional does weigh the state variables by arbitrary even powers. Furthermore, it is interesting to note that Eq. (31) has the form

$$\tilde{J}_{\Delta A}(x_0, u(\cdot)) = \int_0^\infty \left[ x^T \left( R_1 + \sum_{k=2}^r (x^T M_k x)^{k-1} \hat{R}_k \right) x + u^T R_2 u + \phi_{\text{NL}}^T(x) R_2 \phi_{\text{NL}}(x) \right] dt,$$

where  $\phi_{\text{NL}}(x)$  is the nonlinear part of the optimal feedback control

$$\phi(x) = \phi_L(x) + \phi_{\text{NL}}(x),$$

where  $\phi_L(x) \triangleq -R_2^{-1} B^T P x$  and  $\phi_{\text{NL}}(x) \triangleq -R_2^{-1} B^T \sum_{k=2}^r (x^T M_k x)^{k-1} M_k x$ .

Next, we consider the special case in which  $r=2$ . In this case note that if there exist  $P \in \mathbb{P}^{n \times n}$  and  $M_2 \in \mathbb{N}^{n \times n}$  such that

$$0 = A^T P + P A + R_1 + \Omega(P) - P S P$$

and

$$0 = (A - S P)^T M_2 + M_2 (A - S P) + \hat{R}_2 + \Omega(M_2),$$

then Eq. (25), with the performance functional

$$\tilde{J}_{\Delta A}(x_0, u(\cdot)) = \int_0^\infty [x^T R_1 x + u^T R_2 u + (x^T M_2 x)(x^T \hat{R}_2 x) + (x^T M_2 x)^2 (x^T M_2 S M_2 x)] dt,$$

is globally asymptotically stable for all  $x_0 \in \mathbb{R}^n$  with the feedback control law  $u = \phi(x) = -R_2^{-1} B^T (P + (x^T M_2 x) M_2) x$ .

Having established the theoretical basis for our approach, we now assign explicit structure to the set  $\mathcal{A}$  and the bounding function  $\Omega(\cdot)$ . First, the uncertainty set  $\mathcal{A}$  is assumed to be of the form

$$\mathcal{A} \triangleq \left\{ \Delta A \in \mathbb{R}^{n \times n}: \Delta A = \sum_{i=1}^p \sigma_i A_i, \sum_{i=1}^p \sigma_i^2 / \alpha_i^2 \leq 1 \right\}, \quad (35)$$

where for  $i=1, \dots, p$ ,  $A_i \in \mathbb{R}^{n \times n}$  are fixed matrices denoting the structure of the parametric uncertainty,  $\alpha_i$  is a given positive number, and  $\sigma_i$  is an uncertain real parameter. Note that the uncertain parameters  $\sigma_i$  are assumed to lie in a specified ellipsoidal region in  $\mathbb{R}^p$  [5,6]. In this case, as shown in [5,6],  $\Omega(P) = \sum_{i=1}^p (\alpha_i^2 / \alpha) A_i^T P A_i + \alpha P$ , where  $\alpha$  is an arbitrary positive scalar, satisfies Eq. (26) with  $\mathcal{A}$  given by Eq. (35). For the statement of the next result define  $A_\alpha \triangleq A + (\alpha/2) I_n$ .

**Proposition 3.1.** Consider the uncertain system

$$\dot{x}(t) = (A + \Delta A)x(t) + B u(t), \quad x(0) = x_0, \quad t \geq 0, \quad (36)$$

where  $u(\cdot)$  is admissible and  $\Delta A \in \mathcal{A}$  where  $\mathcal{A}$  is given by Eq. (35). Assume there exist  $P \in \mathbb{P}^{n \times n}$  and  $M_2 \in \mathbb{N}^{n \times n}$  such that

$$0 = A_\alpha^T P + P A_\alpha + R_1 + \sum_{i=1}^p \frac{\alpha_i^2}{\alpha} A_i^T P A_i - P S P, \quad (37)$$

$$0 = (A_\alpha - S P)^T M_2 + M_2 (A_\alpha - S P) + \sum_{i=1}^p \frac{\alpha_i^2}{\alpha} A_i^T M_2 A_i + \hat{R}_2. \quad (38)$$

Furthermore, let

$$\tilde{L}(x, u) = x^T [R_1 + (x^T M_2 x) \hat{R}_2 + (x^T M_2 x)^2 M_2 S M_2] x + u^T R_2 u,$$

and

$$\tilde{\Gamma}(x, u) = x^T \left[ \sum_{i=1}^p \frac{\alpha_i^2}{\alpha} A_i^T P A_i + \alpha P + (x^T M_2 x) \left( \sum_{i=1}^p \frac{\alpha_i^2}{\alpha} A_i^T M_2 A_i + \alpha M_2 \right) \right] x.$$

Then the uncertain system (36), with performance functional

$$\tilde{J}_{\Delta A}(x_0, u(\cdot)) \triangleq \int_0^\infty \tilde{L}(x, u) dt, \tag{39}$$

is globally asymptotically stable for all  $x_0 \in \mathbb{R}^n$  and  $\Delta A \in \mathcal{A}$ , with the feedback control  $u = \phi(x) = -R_2^{-1} B^T (P + (x^T M_2 x) M_2) x$ . Furthermore,

$$\sup_{\Delta A \in \mathcal{A}} \tilde{J}_{\Delta A}(x_0, \phi(x(\cdot))) \leq \tilde{\mathcal{J}}(x_0, \phi(x(\cdot))) = x_0^T P x_0 + \frac{1}{2} (x_0^T M_2 x_0)^2, \tag{40}$$

where

$$\tilde{\mathcal{J}}(x_0, u(\cdot)) \triangleq \int_0^\infty [\tilde{L}(x, u) + \tilde{\Gamma}(x, u)] dt, \tag{41}$$

and where  $u(\cdot)$  is admissible, and  $x(t)$ ,  $t \geq 0$ , solves Eq. (36) with  $\Delta A = 0$ . In addition,

$$\tilde{\mathcal{J}}(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} \tilde{\mathcal{J}}(x_0, u(\cdot)), \tag{42}$$

where  $\mathcal{S}(x_0)$  is the set of asymptotically stabilizing controllers for the nominal system and  $x_0 \in \mathbb{R}^n$ .

**Proof.** It need only be noted that  $\Delta A^T P + P \Delta A \leq \sum_{i=1}^p (\alpha_i^2 / \alpha) A_i^T P A_i + \alpha P$  for all  $\Delta A \in \mathcal{A}$  and  $P \in \mathbb{P}^{n \times n}$ . The result now is a direct consequence of Theorem 3.1 with  $\Omega(P) = \sum_{i=1}^p (\alpha_i^2 / \alpha) A_i^T P A_i + \alpha P$  and  $r = 2$ .  $\square$

Next, we assign a different structure to the uncertainty set  $\mathcal{A}$  and the bounding function  $\Omega(\cdot)$ . Specifically, the uncertainty set  $\mathcal{A}$  is assumed to be of the form

$$\mathcal{A} = \{ \Delta A \in \mathbb{R}^{n \times n}: \Delta A = B_0 F C_0, F^T F \leq N \}, \tag{43}$$

where  $B_0 \in \mathbb{R}^{n \times r}$  and  $C_0 \in \mathbb{R}^{r \times n}$  are fixed matrices denoting the structure of the uncertainty,  $F \in \mathbb{R}^{r \times r}$  is an uncertain matrix, and  $N \in \mathbb{N}^{r \times r}$  is a given uncertainty bound [4]. In this case, as shown in [4],  $\Omega(P) = C_0^T N C_0 + P B_0 B_0^T P$  satisfies Eq. (26) with  $\mathcal{A}$  given by Eq. (43).

**Proposition 3.2.** Consider the uncertain system

$$\dot{x}(t) = (A + \Delta A)x(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{44}$$

where  $u(\cdot)$  is admissible and  $\Delta A \in \mathcal{A}$  where  $\mathcal{A}$  is given by Eq. (43). Assume there exist  $P \in \mathbb{P}^{n \times n}$  and  $M_2 \in \mathbb{N}^{n \times n}$  such that

$$0 = A^T P + PA + R_1 + C_0^T N C_0 + P B_0 B_0^T P - P S P, \tag{45}$$

$$0 = (A - S P)^T M_2 + M_2 (A - S P) + \hat{R}_2 + C_0^T N C_0 + M_2 B_0 B_0^T M_2. \tag{46}$$

Furthermore, let

$$\tilde{L}(x, u) = x^T [R_1 + (x^T M_2 x) \hat{R}_2 + (x^T M_2 x)^2 M_2 S M_2] x + u^T R_2 u,$$

and

$$\tilde{\Gamma}(x, u) = x^T [C_0^T N C_0 + P B_0 B_0^T P + (x^T M_2 x)(C_0^T N C_0 + M_2 B_0 B_0^T M_2)] x.$$

Then the uncertain system (44), with performance functional

$$\tilde{J}_{\Delta A}(x_0, u(\cdot)) \triangleq \int_0^\infty \tilde{L}(x, u) dt, \tag{47}$$

is globally asymptotically stable for all  $x_0 \in \mathbb{R}^n$  and  $\Delta A \in \mathcal{A}$ , with the feedback control  $u = \phi(x) = -R_2^{-1} B^T (P + (x^T M_2 x) M_2) x$ . Furthermore,

$$\sup_{\Delta A \in \mathcal{A}} \tilde{J}_{\Delta A}(x_0, \phi(x(\cdot))) \leq \tilde{\mathcal{J}}(x_0, \phi(x(\cdot))) = x_0^T P x_0 + \frac{1}{2} (x_0^T M_2 x_0)^2, \tag{48}$$

where

$$\tilde{\mathcal{J}}(x_0, u(\cdot)) \triangleq \int_0^\infty [\tilde{L}(x, u) + \tilde{\Gamma}(x, u)] dt, \tag{49}$$

and where  $u(\cdot)$  is admissible, and  $x(t)$ ,  $t \geq 0$ , solves Eq. (44) with  $\Delta A = 0$ . In addition,

$$\tilde{\mathcal{J}}(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} \tilde{\mathcal{J}}(x_0, u(\cdot)), \tag{50}$$

where  $\mathcal{S}(x_0)$  is the set of asymptotically stabilizing controllers for the nominal system and  $x_0 \in \mathbb{R}^n$ .

**Proof.** It need only be noted that  $\Delta A^T P + P \Delta A \leq C_0^T N C_0 + P B_0 B_0^T P$  for all  $\Delta A \in \mathcal{A}$  and  $P \in \mathbb{P}^{n \times n}$ . The result now is a direct consequence of Theorem 3.1 with  $\Omega(P) = C_0^T N C_0 + P B_0 B_0^T P$  and  $r = 2$ .  $\square$

**Remark 3.5.** Propositions 3.1 and 3.2 are generalizations of results given in [4, 6] to robust nonlinear control for uncertain linear systems with nonlinear polynomial performance criteria.

#### 4. Illustrative numerical example

In this section we apply the robust nonlinear control design framework to an illustrative numerical example. Specifically, consider the pitch axis longitudinal dynamics model of the F-16 fighter aircraft system given in [23] for nominal flight conditions at 3000 ft and Mach number of 0.6. The nominal model is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1.00 & 0 \\ 0 & -0.87 & 43.22 \\ 0 & 0.99 & -1.34 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -17.25 & -1.58 \\ -0.17 & -0.25 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \tag{51}$$

where  $x_1$  is the pitch angle,  $x_2$  is the pitch rate,  $x_3$  is the angle of attack,  $u_1$  is the elevator deflection, and  $u_2$  is the flaperon deflection. Here we consider uncertainty in the (2,2) and (3,3) components of the dynamics matrix. Using the uncertainty structure given by Eq. (43), the actual dynamics are given by  $A_{\text{actual}} = A + B_0 F C_0$ , where

$$B_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using Proposition 3.2, with  $R_1 = 3I_3$ ,  $R_2 = 0.001I_2$ ,  $\hat{R}_2 = 10R_1$ ,  $|f_1| \leq 1$ ,  $|f_2| \leq 5$ , and  $r = 2$ , a robustified nonlinear controller was designed for the uncertain system. This controller was compared with the robustified linear ( $H_\infty$ ) controller given in [4], the Speyer [25] nonlinear controller, and the LQR controller. Fig. 1 shows that for the case where  $f_1 = -1$  and  $f_2 = 5$  the LQR controller destabilizes the system while the nominal

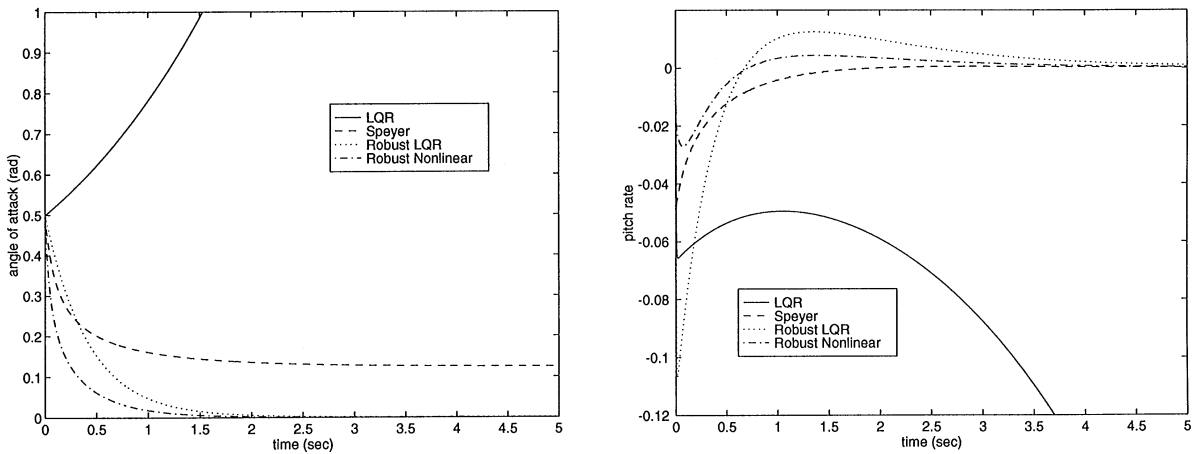


Fig. 1. Comparison of LQR, Speyer [25], Robust LQR, and robust nonlinear controllers for the uncertain system.

Speyer [25] controller maintains Lyapunov stability. This demonstrates the inherent robustness of the nominal (nonrobustified) nonlinear control in comparison to the nominal linear control. Furthermore for the same uncertainty range the figure shows the state response for the robustified nonlinear controller (Proposition 3.2) and the robustified linear ( $H_\infty$ ) controller obtained in [4].

## 5. Conclusion

A unified framework was developed to address the problem of optimal nonlinear-nonquadratic robust control for linear uncertain systems. Specifically, by properly modifying the nonlinear-nonquadratic performance criterion to account for system uncertainty the robust control problem was translated into an optimal control problem that guarantees robust stability and performance for a class of nonlinear uncertain systems. The overall framework provides for a generalization of the classical Hamilton–Jacobi–Bellman conditions for designing robust nonlinear optimal controllers for uncertain systems. The present framework extends the nonlinear–nonquadratic optimal control problems considered in [2, 25] to robust nonlinear optimal control. Finally, a design example was presented to demonstrate the efficacy of robust nonlinear controllers over robust linear controllers for linear uncertain systems.

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