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Optimal nonlinear–nonquadratic feedback control for systems with L_2 and L_∞ disturbances[1]

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1. Introduction

One of the fundamental problems in analysis and feedback control design of linear systems is the ability of the control system to reject uncertain exogenous disturbances. To this end, H_∞ control theory has been developed to address the problem of disturbance rejection for systems with bounded energy (square-integrable) L_2 signal norms on the disturbances and performance variables [1–3]. Since in this case the induced H_∞ transfer function norm corresponds to the worst-case disturbance attenuation, for systems with poorly modeled disturbances which possess significant power within arbitrarily small bandwidths H_∞ theory is clearly appropriate. For linear finite-dimensional time-invariant systems the H_∞ control design problem has been thoroughly investigated in recent years (see, for example [4–7] and the numerous references therein). In particular, the H_∞ control design problem was formulated in the state space setting and was shown to correspond to a two-person zero sum differential game problem wherein the existence of an H_∞ (sub)optimal controller is equivalent to the existence of a solution to an algebraic Riccati equation arising in quadratic differential game theory [8–10].

Alternatively, the H_∞ analysis and synthesis problem can also be formulated and solved in the state space setting using the notion of dissipativity theory [11]. In particular, using the bounded real Riccati equation it follows that H_∞ norm of a (closed-loop) linear system is less than a prespecified positive number γ if and only if the (closed-

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loop) linear system is nonexpansive (gain bounded) with respect to an appropriate quadratic supply rate involving the systems weighted input energy and output energy. Riccati equation based results for the mixed-norm H_2/H_∞ problem [12–14] have also been developed using dissipativity notions to allow the trade-off between systems with stochastic white noise disturbance models (H_2 models) possessing a fixed covariance (power spectral density) and deterministic bounded energy disturbance models. As in the pure H_∞ case the mixed-norm H_2/H_∞ problem can be compared to a game-theoretic framework involving a Nash differential game problem [15].

Using a nonlinear game-theoretic framework the authors in [10, 16] replace the algebraic Riccati equation arising in linear H_∞ theory with a particular Hamilton–Jacobi–Bellman equation (the Isaacs equation) to obtain a nonlinear equivalent to the H_∞ analysis and synthesis control problem. Sufficient conditions for the existence of stabilizing solutions of the Isaacs equation are given in [17–19] in terms of the existence of a linear (sub)optimal H_∞ controller for the linearized (about an equilibrium point) nonlinear controlled system. In parallel research the authors in [20–23] use nonlinear dissipation theory [24–30] for nonlinear affine systems with appropriate storage functions and quadratic supply rates to obtain nonexpansive (gain bounded) closed-loop systems.

Although a nonlinear equivalent to H_∞ analysis and synthesis has been developed it is important to note that the methods and results discussed in [10, 16–23] are independent of optimality considerations. In this paper we develop an optimality-based theory for disturbance rejection for nonlinear systems with bounded exogenous disturbances. The key motivation for developing an optimal nonlinear control theory that additionally guarantees disturbance rejection is that it provides a class of candidate disturbance rejection controllers parameterized by the cost functional that is minimized. In the case of linear systems optimality-based theories have proven extremely successful in numerous applications. Specifically, to fully address the trade-offs between H_2 and H_∞ performance the optimality-based linear-quadratic control problem was merged with H_∞ methods to address the mixed-norm H_2/H_∞ control problem [12, 13].

In order to address the optimality-based disturbance rejection nonlinear control problem we extend the nonlinear–nonquadratic controller analysis and synthesis framework developed in [31]. The basic underlying ideas of results in [31] are based on the fact that the steady-state solution of the Hamilton–Jacobi–Bellman equation is a Lyapunov function for the nonlinear controlled system thus guaranteeing both optimality and stability [32]. In this paper we extend the framework developed in [31] to address the problem of optimal nonlinear–nonquadratic control that additionally guarantees disturbance rejection to bounded input disturbances. Specifically, using nonlinear dissipation theory with appropriate storage functions and supply rates we transform the nonlinear disturbance rejection problem into an optimal control problem. This is accomplished by properly modifying the cost functional to account for exogenous disturbances so that the solution of the modified optimal nonlinear control problem serves as the solution to the disturbance rejection problem.

Our framework guarantees that the closed-loop nonlinear input–output map is dissipative with respect to general supply rates. Specializing to quadratic supply rates involving net system energy flow and weighted input and output energy our results

guarantee passive and nonexpansive (gain bounded) closed-loop input–output maps, respectively. In the special case where the controlled system is linear our results, with appropriate quadratic supply rates, specialize to the mixed-norm H_2/H_∞ framework developed in [12] and the mixed H_2/positivity framework developed in [33, 34].

The main contribution of this paper is a methodology for designing optimal nonlinear controllers which guarantee disturbance rejection and minimize a performance functional that serves as an upper bound to a nonlinear–nonquadratic cost functional. In particular, the performance bound can be evaluated in closed-form as long as the nonlinear–nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees stability. This Lyapunov function is shown to be the solution to the steady-state form of the Hamilton–Jacobi–Bellman equation for the controlled system and plays a key role in constructing the optimal nonlinear control law. Furthermore, since our nonlinear–nonquadratic cost functional is closely related to the structure of the Lyapunov function the proposed framework provides a class of feedback stabilizing controllers that minimize a specified performance functional. Hence the overall framework provides for a generalization of the Hamilton–Jacobi–Bellman conditions for addressing the design of optimal controllers for nonlinear systems with exogenous disturbances.

A key feature of the present work is that since the necessary Hamilton–Jacobi–Bellman optimality conditions are obtained for a modified nonlinear–nonquadratic performance functional rather than the original performance functional, *globally* optimal controllers are guaranteed to provide disturbance rejection. Of course, since our approach allows us to construct globally optimal controllers that minimize a given Hamiltonian, the resulting disturbance rejection controllers provide the best worst-case performance over the class of input disturbances.

The contents of the paper are as follows. In Section 2 we establish notation and mathematical preliminaries. In Section 3 we consider a nonlinear system with bounded input disturbances and a nonlinear–nonquadratic performance functional evaluated over the infinite horizon. The performance functional is then evaluated in terms of a Lyapunov function that guarantees stability and dissipativity with respect to general supply rates. This result is then specialized to linear systems with quadratic supply rates and connections are drawn to the mixed-norm H_2/H_∞ and mixed H_2/positivity linear-quadratic problems developed in [12] and [33, 34], respectively. Furthermore, as a special case of the framework developed in Section 3 we provide a Riccati equation characterization for capturing the L_1 norm (L_∞ equi-induced norm) of a linear time-invariant system. It is further shown that this framework can be used to enforce point-wise-in-time performance constraints (L_1 performance) providing worst-case amplification attenuation from L_∞ input–output signals along with H_2 performance. In Section 4 we state a nonlinear disturbance rejection control problem and transform the problem to an optimal control problem. Theorem 4.1 presents sufficient conditions that characterize an optimal nonlinear feedback controller that guarantees stability and dissipativity of the closed-loop system with respect to general supply rates. This result is then specialized to the linear-quadratic case to draw connections with the mixed-norm H_2/H_∞ linear-quadratic regulator problem [12, 35] and the mixed H_2/positivity control problem [33, 34, 36]. In Section 5 we specialize the results of Section 4 to linear systems

controlled by nonlinear controllers subjected to bounded exogenous disturbances that minimize a *multilinear* cost functional. This result generalizes the classic results of Bass and Webber [37] and Speyer [38] to optimal nonlinear control for disturbance rejection of exogenous disturbances. In Section 6 we close the paper with conclusions.

2. Notation and mathematical preliminaries

In this section we establish definitions, notation, and a key lemma used later in the paper. Let $\mathbb{R}$ denote real numbers, let $\mathbb{R}^{n\times m}$ denote real $n\times m$ matrices, and let $\mathbb{N}^{n\times n}$ ($\mathbb{P}^{n\times n}$) denote $n\times n$ nonnegative (positive) definite matrices. Furthermore, $A\geq 0$ ($A>0$) denotes the fact that the Hermitian matrix A is nonnegative (positive) definite and $A\geq B$ ($A>B$) denotes the fact that $A-B\geq 0$ ($A-B>0$). $A\otimes B$ and $A\oplus B$ denote the Kronecker product and the Kronecker sum, respectively, of matrices A and B. Let L_2 (resp., L_∞) denote square-integrable (resp., bounded) Lebesgue measurable functions on $[0,\infty)$ and let H_2 (resp., H_∞) denote the Hardy space of real-rational transfer function matrices square-integrable (resp., bounded) on the imaginary axis with analytic continuation into the right-half plane. A scalar function $\psi:\mathbb{R}^n\to\mathbb{R}$ is *k-multilinear* if k is a positive integer and $\psi(x)$ is a linear combination of terms of the form $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$, where i_j is a nonnegative integer for $j=1,\ldots,n$, and $i_1+i_2+\cdots+i_n=k$. Furthermore, a k-multilinear function $\psi(\cdot)$ is *nonnegative definite* (resp., *positive definite*) if $\psi(x)\geq 0$ for all $x\in\mathbb{R}^n$ (resp., $\psi(x)>0$ for all nonzero $x\in\mathbb{R}^n$). Note that if k is odd then $\psi(x)$ cannot be positive definite. If $\psi(\cdot)$ is a k-multilinear function then $\psi(\cdot)$ can be represented by means of Kronecker products, that is, $\psi(x)$ is given by $\psi(x)=\Psi x^{[k]}$ where $\Psi\in\mathbb{R}^{1\times n^k}$ and $x^{[k]}\triangleq\otimes x\otimes\cdots\otimes x$ (k times). Next, define the repeated (k times) Kronecker sum as $\oplus^k A\triangleq A\oplus A\oplus\cdots\oplus A$, and let $(\ )'$ denote the Frechét derivative. The following lemma is needed for one of the main results of this paper.

Lemma 2.1 [31]. *Let $A\in\mathbb{R}^{n\times n}$ be asymptotically stable and let $h:\mathbb{R}^n\to\mathbb{R}$ be a k-multilinear function. Then there exists a unique k-multilinear function $g:\mathbb{R}^n\to\mathbb{R}$ such that*

$$0=g'(x)Ax+h(x),\quad x\in\mathbb{R}^n. \tag{1}$$

Furthermore, if $h(x)$ is nonnegative (resp., positive) definite, then $g(x)$ is nonnegative (resp., positive) definite.

In this paper we consider nonlinear systems $\mathscr{G}$ of the form

$$\dot{x}(t)=f(x(t))+J_1(x(t))w(t),\quad x(0)=x_0,\quad t\geq 0, \tag{2}$$

$$z(t)=h(x(t))+J_2(x(t))w(t), \tag{3}$$

where $x\in\mathbb{R}^n$, $w\in\mathbb{R}^d$, $z\in\mathbb{R}^p$, $f:\mathbb{R}^n\to\mathbb{R}^n$, $J_1:\mathbb{R}^n\to\mathbb{R}^{n\times d}$, $h:\mathbb{R}^n\to\mathbb{R}^p$, and $J_2:\mathbb{R}^n\to\mathbb{R}^{l\times d}$. We assume that $f(\cdot)$, $J_1(\cdot)$, $h(\cdot)$, and $J_2(\cdot)$ are smooth (C^∞ mappings) and $f(\cdot)$ has at least one equilibrium so that, after possibly a coordinate transformation, $f(0)=0$ and $h(0)=0$. Furthermore, for the nonlinear system $\mathscr{G}$ we assume that the

required properties for the existence and uniqueness of solutions are satisfied, that is, $w(\cdot)$ satisfies sufficient regularity conditions such that (2) has a unique solution forward in time. For the dynamical system $\mathscr{G}$ given by (2) and (3) a function $r:\mathbb{R}^p\times\mathbb{R}^d\to\mathbb{R}$ is called a *supply rate* [24] if it is locally integrable, that is, for all input–output pairs $w\in\mathbb{R}^d$, $z\in\mathbb{R}^p$, it satisfies $\int_{t_1}^{t_2}|r(z(s),w(s))|\,\mathrm{d}s<\infty$, $t_1,t_2\geq 0$.

Definition 2.1 [24]. A system $\mathscr{G}$ of the form (2), (3) is *dissipative with respect to the supply rate r* if there exists a C^0 nonnegative-definite function $V:\mathbb{R}^n\to\mathbb{R}$, called a *storage function*, such that the *dissipation inequality*

$$V(x(t))\leq V(x(t_1))+\int_{t_1}^{t} r(z(s),w(s))\,\mathrm{d}s, \tag{4}$$

is satisfied for all $t\geq t_1\geq 0$ and where $x(t)$, $t\geq 0$, is the solution to (2) with $w\in\mathbb{R}^d$.

Definition 2.2 [11]. A system $\mathscr{G}$ of the form (2), (3) with $x(0)=x_0$ is *nonexpansive* if the solution $x(t)$, $t\geq 0$, to (2) satisfies

$$\int_0^t z^{\mathrm{T}}(s)z(s)\,\mathrm{d}s\leq\gamma^2\int_0^t w^{\mathrm{T}}(s)w(s)\,\mathrm{d}s+V(x_0), \tag{5}$$

for all $t\geq 0$, $w(\cdot)\in\mathrm{L}_2$, and where $\gamma>0$ is given.

Definition 2.3 [11]. A system $\mathscr{G}$ of the form (2), (3) with $x(0)=x_0$ and $p=d$ is *passive* if the solution $x(t)$, $t\geq 0$, to (2) satisfies

$$\int_0^t 2z^{\mathrm{T}}(s)w(s)\,\mathrm{d}s+V(x_0)\geq 0, \tag{6}$$

for all $t\geq 0$ and $w(\cdot)\in\mathrm{L}_2$.

Note that appropriate supply rates for testing nonexpansivity and passivity of $\mathscr{G}$ are $r(z,w)=\gamma^2w^{\mathrm{T}}w-z^{\mathrm{T}}z$ and $r(z,w)=2z^{\mathrm{T}}w$, respectively.

3. Dissipative nonlinear systems with bounded disturbances

In this section we present sufficient conditions for dissipativity for a class of nonlinear systems with bounded energy and bounded amplitude disturbances. In addition we consider the problem of evaluating a performance bound for a nonlinear–nonquadratic cost functional. The cost bound is evaluated in closed-form by relating the cost functional to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear system.

In this paper we restrict our attention to time-invariant infinite horizon systems. For the following result, let $\mathscr{D}\subset\mathbb{R}^n$ be an open set, assume $0\in\mathscr{D}$, let $L:\mathscr{D}\to\mathbb{R}$, let $f:\mathscr{D}\to\mathbb{R}^n$ such that $f(0)=0$, and let $h:\mathscr{D}\to\mathbb{R}^p$, $J_1:\mathscr{D}\to\mathbb{R}^{n\times d}$, and $J_2:\mathscr{D}\to\mathbb{R}^{p\times d}$. Finally, let $\mathscr{W}\subset\mathbb{R}^d$ be a subset of measurable functions and let $r:\mathbb{R}^p\times\mathbb{R}^d\to\mathbb{R}$ be a given function.

First we present sufficient conditions under which a nonlinear system is dissipative with respect to the supply rate $r(z,w)$.

Lemma 3.1. *Consider the nonlinear system*

$$\dot{x}(t) = f(x(t)) + J_1(x(t))w(t), \quad x(0) = x_0, \ t \geq 0, \ w(\cdot) \in \mathscr{W}, \tag{7}$$

$$z(t) = h(x(t)) + J_2(x(t))w(t). \tag{8}$$

Furthermore, assume there exist functions $\Gamma : \mathscr{D} \to \mathbb{R}$ *and* $V : \mathscr{D} \to \mathbb{R}$ *where* $V(\cdot)$ *is a* C^1 *nonnegative-definite function such that*

$$V(0) = 0, \tag{9}$$

$$V'(x)J_1(x)w \leq r(z,w) + \Gamma(x), \quad x \in \mathscr{D}, \ w \in \mathbb{R}^d, \tag{10}$$

$$V'(x)f(x) + \Gamma(x) \leq 0, \quad x \in \mathscr{D}. \tag{11}$$

Then the solution $x(t)$, $t \geq 0$, *of* (7) *satisfies*

$$V(x(T)) \leq \int_0^T r(z(t), w(t))\mathrm{d}t + V(x_0), \quad T \geq 0, \ w(\cdot) \in \mathscr{W}. \tag{12}$$

Proof. Let $x(t)$, $t \geq 0$, satisfy (7) and let $w(\cdot) \in \mathscr{W}$. Then it follows from (10) and (11) that

$$\begin{aligned} \dot{V}(x(t)) &\triangleq \frac{\mathrm{d}V(x(t))}{\mathrm{d}t} = V'(x(t))[f(x(t)) + J_1(x(t))w(t)] \\ &\leq V'(x(t))f(x(t)) + \Gamma(x(t)) + r(z(t), w(t)) \\ &\leq r(z(t), w(t)). \end{aligned} \tag{13}$$

Next, integrating over $[0, T]$ yields

$$V(x(T)) - V(x_0) \leq \int_0^T r(z(t), w(t))\,\mathrm{d}t, \quad T \geq 0. \qquad \square$$

For the next result let $L : \mathscr{D} \to \mathbb{R}$ be given.

Theorem 3.1. *Consider the nonlinear system given by* (7), (8) *with performance functional*

$$J(x_0) \triangleq \int_0^\infty L(x(t))\,\mathrm{d}t, \tag{14}$$

where $x(t)$, $t \geq 0$, *solves* (7) *with* $w(t) \equiv 0$. *Assume there exist functions* $\Gamma : \mathscr{D} \to \mathbb{R}$ *and* $V : \mathscr{D} \to \mathbb{R}$ *where* $V(\cdot)$ *is* C^1 *such that*

$$V(0) = 0, \tag{15}$$

$$V(x) > 0, \quad x \in \mathscr{D}, \ x \neq 0, \tag{16}$$

$$V'(x)f(x)<0, \quad x\in\mathscr{D},\ x\neq 0, \tag{17}$$

$$V'(x)J_1(x)w\leq r(z,w)+L(x)+\Gamma(x), \quad x\in\mathscr{D},\ w\in \mathrm{L}_2, \tag{18}$$

$$L(x)+V'(x)f(x)+\Gamma(x)=0, \quad x\in\mathscr{D}. \tag{19}$$

Then there exists a neighborhood $\mathscr{D}_0\subset\mathscr{D}$ of the origin such that if $x_0\in\mathscr{D}_0$, then $x(t)\equiv 0$ is a locally asymptotically stable solution of (7) *with $w(t)\equiv 0$. If, in addition, $\Gamma(x)\geq 0$, $x\in\mathscr{D}$, then*

$$J(x_0)\leq \mathscr{J}(x_0)=V(x_0), \tag{20}$$

where

$$\mathscr{J}(x_0)\triangleq\int_0^\infty [L(x(t))+\Gamma(x(t))]\,\mathrm{d}t \tag{21}$$

and where $x(t)$, $t\geq 0$, is a solution to (7) *with $w(t)\equiv 0$. Furthermore, the solution $x(t)$, $t\geq 0$, of* (7) *satisfies the dissipativity constraint*

$$\int_0^T r(z(t),w(t))\,\mathrm{d}t+V(x_0)\geq 0, \quad T\geq 0,\ w(\cdot)\in \mathrm{L}_2. \tag{22}$$

Finally, if $\mathscr{D}=\mathbb{R}^n$, $w(t)\equiv 0$, and

$$V(x)\to\infty \quad \text{as } \|x\|\to\infty, \tag{23}$$

then the solution $x(t)=0$, $t\geq 0$, of (7) *is globally asymptotically stable.*

Proof. Let $x(t)$, $t\geq 0$, satisfy (7). Then

$$\dot V(x(t))\triangleq\frac{\mathrm{d}}{\mathrm{d}t}V(x(t))=V'(x(t))(f(x(t))+J_1(x)w(t)), \quad t\geq 0. \tag{24}$$

Hence, with $w(t)\equiv 0$, it follows from (17) that

$$\dot V(x(t))<0, \quad t\geq 0,\ x(t)\neq 0. \tag{25}$$

Thus, from (15), (16), and (25) it follows that $V(\cdot)$ is a Lyapunov function [39] for (7), which proves local asymptotic stability of the solution $x(t)\equiv 0$ with $w(t)\equiv 0$. Consequently, $x(t)\to 0$ as $t\to\infty$ for all initial conditions $x_0\in\mathscr{D}_0$ for some neighborhood $\mathscr{D}_0\subset\mathscr{D}$ of the origin. Now for $w(t)\equiv 0$, (25) implies that

$$0=-\dot V(x(t))+V'(x(t))f(x(t)), \quad t\geq 0.$$

Next, if $\Gamma(x)\geq 0$, $x\in\mathscr{D}$, (19) implies

$$\begin{aligned}L(x(t))&=-\dot V(x(t))+L(x(t))+V'(x(t))f(x(t))\\&\leq -\dot V(x(t))+L(x(t))+V'(x(t))f(x(t))+\Gamma(x(t))\\&=-\dot V(x(t)).\end{aligned}$$

Now, integrating over $[0,t)$ yields

$$\int_0^t L(x(s))\,\mathrm{d}s \le -V(x(t)) + V(x_0).$$

Letting $t\to\infty$ and noting that $V(x(t))\to 0$ for all $x_0\in\mathscr{D}_0$ yields $J(x_0)\le V(x_0)$. Next, let $x(t)$, $t\ge 0$, satisfy (7) with $w(t)\equiv 0$. Then it follows from (19) that

$$\begin{aligned} L(x(t)) + \Gamma(x(t)) &= -\dot{V}(x(t)) + L(x(t)) + V'(x(t))f(x(t)) + \Gamma(x(t)) \\ &= -\dot{V}(x(t)). \end{aligned}$$

Integrating over $[0,t)$ yields

$$\int_0^t [L(x(s)) + \Gamma(x(s))]\,\mathrm{d}s = -V(x(t)) + V(x_0).$$

Now letting $t\to\infty$ yields $\mathscr{J}(x_0) = V(x_0)$. Finally, it follows that (15), (18), and (19) imply (9)–(11) and hence, with $\Gamma(x)$ replaced by $L(x)+\Gamma(x)$ and $\mathscr{W} = \mathrm{L}_2$, Lemma 3.1 yields

$$V(x(T)) \le \int_0^T r(z(t),w(t))\,\mathrm{d}t + V(x_0), \quad w(\cdot)\in \mathrm{L}_2,\ T\ge 0.$$

Now (22) follows by noting that $V(x(T))\ge 0$. Finally, for $\mathscr{D} = \mathbb{R}^n$ global asymptotic stability of the solution $x(t)=0$, $t\ge 0$, is a direct consequence of the radially unbounded condition (23) on $V(x)$. □

3.1. Specialization to dissipative systems with quadratic supply rates

In this section we consider the special case in which $r(z,w)$ is a quadratic functional. Specifically, let $h:\mathscr{D}\to\mathbb{R}^p$, $J_2:\mathscr{D}\to\mathbb{R}^{p\times d}$, $\hat{Q}\in\mathbb{R}^{p\times p}$, $\hat{S}\in\mathbb{R}^{p\times d}$, $\hat{R}\in\mathbb{R}^{d\times d}$, and

$$r(z,w) = z^{\mathrm{T}}\hat{Q}z + 2z^{\mathrm{T}}\hat{S}w + w^{\mathrm{T}}\hat{R}w, \tag{26}$$

such that

$$N(x) \triangleq J_2^{\mathrm{T}}(x)\hat{Q}J_2(x) + J_2^{\mathrm{T}}(x)\hat{S} + \hat{S}^{\mathrm{T}}J_2(x) + \hat{R} > 0, \quad x\in\mathscr{D}.$$

Furthermore, let $L(x)\ge 0$, $x\in\mathscr{D}$. Then

$$\begin{aligned} \Gamma(x) = {} & [\tfrac{1}{2}J_1^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) - J_2^{\mathrm{T}}(x)\hat{Q}h(x) - \hat{S}^{\mathrm{T}}h(x)]^{\mathrm{T}}N^{-1}(x) \\ & \cdot [\tfrac{1}{2}J_1^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) - J_2^{\mathrm{T}}(x)\hat{Q}h(x) - \hat{S}^{\mathrm{T}}h(x)] - h^{\mathrm{T}}(x)\hat{Q}h(x), \end{aligned}$$

satisfies (18) since in this case

$$\begin{aligned} & L(x) + \Gamma(x) - V'(x)J_1(x) + r(z,w) \\ & \quad = L(x) + [\tfrac{1}{2}J_1^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) - J_2^{\mathrm{T}}(x)\hat{Q}(x)h(x) - \hat{S}^{\mathrm{T}}(x)h(x) - N(x)w]^{\mathrm{T}}N^{-1}(x) \end{aligned}$$

$$\cdot\,[\tfrac{1}{2}J_1^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) - J_2^{\mathrm{T}}(x)\hat{Q}(x)h(x) - \hat{S}^{\mathrm{T}}(x)h(x) - N(x)w]$$
$$\geq 0. \tag{27}$$

Corollary 3.1. *Let $L(x)\geq 0$, $x\in\mathscr{D}$, and consider the nonlinear system given by* (7), (8) *with performance functional*

$$J(x_0)\triangleq\int_0^\infty L(x(t))\,\mathrm{d}t, \tag{28}$$

where $x(t)$, $t\geq 0$, solves (7) *with $w(t)\equiv 0$. Assume there exists a function $V:\mathscr{D}\to\mathbb{R}$ where $V(\cdot)$ is C^1 such that*

$$V(0)=0, \tag{29}$$

$$V(x)>0,\quad x\in\mathscr{D},\ x\neq 0, \tag{30}$$

$$V'(x)f(x)<0,\quad x\in\mathscr{D},\ x\neq 0, \tag{31}$$

$$L(x)+V'(x)f(x)+\Gamma(x)=0,\quad x\in\mathscr{D}, \tag{32}$$

where

$$\Gamma(x)=[\tfrac{1}{2}J_1^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)+J_2^{\mathrm{T}}(x)h(x)]^{\mathrm{T}}[\gamma^2 I-J_2^{\mathrm{T}}(x)J_2(x)]^{-1}$$
$$\times[\tfrac{1}{2}J_1^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)+J_2^{\mathrm{T}}(x)h(x)]+h^{\mathrm{T}}(x)h(x). \tag{33}$$

Then there exists a neighborhood $\mathscr{D}_0\subset\mathscr{D}$ of the origin such that if $x_0\in\mathscr{D}_0$, then $x(t)\equiv 0$ is a locally asymptotically stable solution of (7) *with $w(t)\equiv 0$. Furthermore,*

$$J(x_0)\leq\mathscr{J}(x_0)=V(x_0), \tag{34}$$

where

$$\mathscr{J}(x_0)\triangleq\int_0^\infty[L(x(t))+\Gamma(x(t))]\,\mathrm{d}t \tag{35}$$

and where $x(t)$, $t\geq 0$, solves (7) *with $w(t)\equiv 0$. Furthermore, the solution $x(t)$, $t\geq 0$, of* (7) *satisfies the nonexpansivity constraint*

$$\int_0^T z(t)^{\mathrm{T}}z(t)\,\mathrm{d}t\leq\gamma^2\int_0^T w^{\mathrm{T}}(t)w(t)\,\mathrm{d}t+V(x_0),\quad T\geq 0,\ w(\cdot)\in\mathrm{L}_2. \tag{36}$$

Finally, if $\mathscr{D}=\mathbb{R}^n$, $w(t)\equiv 0$, and

$$V(x)\to\infty\quad\text{as}\ \|x\|\to\infty, \tag{37}$$

then the solution $x(t)=0$, $t\geq 0$, of (7) *is globally asymptotically stable.*

Proof. With $\hat{Q}=-I$, $\hat{S}=0$, and $\hat{R}=\gamma^2 I$, it follows from (27) that $\Gamma(x)$ given by (33) satisfies (18). The result now follows as a direct consequence of Theorem 3.1. □

Remark 3.1. Note that if $L(x)=h^{\mathrm{T}}(x)h(x)$ in Corollary 3.1 then $\Gamma(x)$ can be chosen as

$$\Gamma(x)=[\tfrac{1}{2}J_1^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)+J_2^{\mathrm{T}}(x)h(x)]^{\mathrm{T}}[\gamma^2 I-J_2^{\mathrm{T}}(x)J_2(x)]^{-1}$$
$$\times[\tfrac{1}{2}J_1^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)+J_2^{\mathrm{T}}(x)h(x)].$$

Note that the framework presented in Corollary 3.1 is an extension of the mixed-norm $\mathrm{H}_2/\mathrm{H}_\infty$ framework of Bernstein and Haddad [12] to nonlinear systems. Specifically, letting $f(x)=Ax$, $J_1(x)=D$, $h(x)=Ex$, $J_2(x)=0$, $L(x)=x^{\mathrm{T}}Rx$, and $V(x)=x^{\mathrm{T}}Px$ where $A\in\mathbb{R}^{n\times n}$, $D\in\mathbb{R}^{n\times d}$, $E\in\mathbb{R}^{p\times n}$, $R\triangleq E^{\mathrm{T}}E>0$, and $P\in\mathbb{P}^{n\times n}$ satisfies

$$0=A^{\mathrm{T}}P+PA+\gamma^{-2}PDD^{\mathrm{T}}P+R, \tag{38}$$

it follows from Remark 3.1 that $\Gamma(x)$ can be chosen as $\Gamma(x)=\gamma^{-2}x^{\mathrm{T}}PDD^{\mathrm{T}}Px$ and hence Corollary 3.1 with $x_0=0$ implies that

$$\int_0^T x^{\mathrm{T}}(t)Rx(t)\,\mathrm{d}t\le\gamma^2\int_0^T w^{\mathrm{T}}(t)w(t)\,\mathrm{d}t,\quad T\ge 0,\ w(\cdot)\in\mathrm{L}_2, \tag{39}$$

or, equivalently, the H_∞ norm of $G(s)\sim\left[\begin{array}{c|c}A&D\\\hline E&0\end{array}\right]$ satisfies

$$\|G\|_\infty\triangleq\sup_{\omega\in\mathbb{R}}\sigma_{\max}(G(\jmath\omega))\le\gamma, \tag{40}$$

where $\sigma_{\max}(\cdot)$ denotes the maximum singular value. Now, (34) implies

$$\int_0^\infty x^{\mathrm{T}}(t)Rx(t)\,\mathrm{d}t\le\int_0^\infty x^{\mathrm{T}}(t)(R+\gamma^{-2}PDD^{\mathrm{T}}P)x(t)\,\mathrm{d}t$$
$$=\int_0^\infty x_0^{\mathrm{T}}\mathrm{e}^{A^{\mathrm{T}}t}(R+\gamma^{-2}PDD^{\mathrm{T}}P)\mathrm{e}^{At}x_0\,\mathrm{d}t,$$

where $x(t)$, $t\ge 0$, solves (7) with $w(t)\equiv 0$. Now, as is common practice [40], we eliminate explicit dependence on the initial condition x_0 by assuming $x_0x_0^{\mathrm{T}}$ has expected value V, that is, $\mathbb{E}[x_0x_0^{\mathrm{T}}]=V$, where $\mathbb{E}$ denotes expectation. Invoking this step leads to

$$\mathbb{E}\left[\int_0^\infty x^{\mathrm{T}}(t)Rx(t)\,\mathrm{d}t\right]=\mathbb{E}\left[\int_0^\infty x_0^{\mathrm{T}}\mathrm{e}^{A^{\mathrm{T}}t}R\mathrm{e}^{At}x_0\,\mathrm{d}t\right]=\mathbb{E}[x_0^{\mathrm{T}}\hat{P}x_0]=\operatorname{tr}\hat{P}V,$$

where

$$0=A^{\mathrm{T}}\hat{P}+\hat{P}A+R$$

and

$$\mathbb{E}\left[\int_0^\infty x^{\mathrm{T}}(t)(R+\gamma^{-2}PDD^{\mathrm{T}}P)x(t)\,\mathrm{d}t\right]=\mathbb{E}\left[\int_0^\infty x_0^{\mathrm{T}}\mathrm{e}^{A^{\mathrm{T}}t}(R+\gamma^{-2}PDD^{\mathrm{T}}P)\mathrm{e}^{At}x_0\,\mathrm{d}t\right]$$
$$=\mathbb{E}[x_0^{\mathrm{T}}Px_0]=\operatorname{tr}PV,$$

where P satisfies (38). Hence $\|G\|_2^2 = \operatorname{tr} \hat{P}V \le \operatorname{tr} PV$ which implies that $J(x_0)$ given by (28) provides an upper bound to the H_2 norm of $G(s)$.

Corollary 3.2. *Let $L(x) \ge 0$, $x \in \mathscr{D}$, $p = d$, and consider the nonlinear system given by* (7), (8) *with performance functional*

$$J(x_0) \triangleq \int_0^\infty L(x(t))\,\mathrm{d}t, \tag{41}$$

where $x(t)$, $t \ge 0$, solves (7) *with $w(t) \equiv 0$. Assume there exists a function $V : \mathscr{D} \to \mathbb{R}$ where $V(\cdot)$ is C^1 such that*

$$V(0) = 0, \tag{42}$$

$$V(x) > 0, \quad x \in \mathscr{D},\ x \ne 0, \tag{43}$$

$$V'(x) f(x) < 0, \quad x \in \mathscr{D},\ x \ne 0, \tag{44}$$

$$L(x) + V'(x) f(x) + \Gamma(x) = 0, \quad x \in \mathscr{D}, \tag{45}$$

where

$$\Gamma(x) = [\tfrac{1}{2} J_1^{\mathrm{T}}(x) V'^{\mathrm{T}}(x) - h(x)]^{\mathrm{T}} [J_2(x) + J_2^{\mathrm{T}}(x)]^{-1} [\tfrac{1}{2} J_1^{\mathrm{T}}(x) V'^{\mathrm{T}}(x) - h(x)]. \tag{46}$$

Then there exists a neighborhood $\mathscr{D}_0 \subset \mathscr{D}$ of the origin such that if $x_0 \in \mathscr{D}_0$, then $x(t) \equiv 0$ is a locally asymptotically stable solution of (7) *with $w(t) \equiv 0$. Furthermore,*

$$J(x_0) \le \mathscr{J}(x_0) = V(x_0), \tag{47}$$

where

$$\mathscr{J}(x_0) \triangleq \int_0^\infty [L(x(t)) + \Gamma(x(t))]\,\mathrm{d}t \tag{48}$$

and where $x(t)$, $t \ge 0$, solves (7) *with $w(t) \equiv 0$. Furthermore, the solution $x(t)$, $t \ge 0$, of* (7) *satisfies the passivity constraint*

$$\int_0^{\mathrm{T}} 2z^{\mathrm{T}}(t) w(t)\,\mathrm{d}t + V(x_0) \ge 0, \quad T \ge 0,\ w(\cdot) \in \mathrm{L}_2. \tag{49}$$

Finally, if $\mathscr{D} = \mathbb{R}^n$, $w(t) \equiv 0$, and

$$V(x) \to \infty \quad \text{as } \|x\| \to \infty, \tag{50}$$

then the solution $x(t) = 0$, $t \ge 0$, of (7) *is globally asymptotically stable.*

Proof. With $\hat{Q} = 0$, $\hat{S} = I$, and $\hat{R} = 0$, it follows from (27) that $\Gamma(x)$ given by (46) satisfies (18). The result now follows as a direct consequence of Theorem 3.1. □

Note that the framework presented in Corollary 3.2 is an extension of the H_2/positivity framework of Haddad and Bernstein [33, 34] to nonlinear systems. Specifically, letting $f(x) = Ax$, $J_1(x) = D$, $h(x) = Ex$, $J_2(x) = E_\infty$, $L(x) = x^{\mathrm{T}} R x$, $V(x) = x^{\mathrm{T}} P x$,

and $\Gamma(x)=x^{\mathrm{T}}[D^{\mathrm{T}}P-E]^{\mathrm{T}}(E_\infty+E_\infty^{\mathrm{T}})^{-1}[D^{\mathrm{T}}P-E]x$ where $A\in\mathbb{R}^{n\times n}$, $D\in\mathbb{R}^{n\times d}$, $E\in\mathbb{R}^{d\times n}$, $E_\infty\in\mathbb{R}^{d\times d}$, $R\in\mathbb{P}^{n\times n}$, and $P\in\mathbb{P}^{n\times n}$ satisfies

$$0=A^{\mathrm{T}}P+PA+(D^{\mathrm{T}}P-E)^{\mathrm{T}}(E_\infty+E_\infty^{\mathrm{T}})^{-1}(D^{\mathrm{T}}P-E)+R, \tag{51}$$

with $x_0=0$ it follows from Corollary 3.2 that

$$\int_0^T w^{\mathrm{T}}(t)z(t)\,\mathrm{d}t\geq 0,\quad T\geq 0,\ w(\cdot)\in \mathrm{L}_2, \tag{52}$$

or, equivalently,

$$G_\infty(s)+G_\infty^*(s)\geq 0,\qquad \mathrm{Re}[s]>0, \tag{53}$$

where $G_\infty(s)\sim\left[\begin{array}{c|c} A & D\\ \hline E & E_\infty\end{array}\right]$. Now, using similar arguments as in the H_∞ case (47) implies

$$\begin{aligned}\operatorname{tr}\hat{P}V&=\mathbb{E}\left[\int_0^\infty x^{\mathrm{T}}(t)Rx(t)\,\mathrm{d}t\right]\\ &\leq\mathbb{E}\left[\int_0^\infty x^{\mathrm{T}}(t)[R+(D^{\mathrm{T}}P-E)^{\mathrm{T}}(E_\infty+E_\infty^{\mathrm{T}})^{-1}(D^{\mathrm{T}}P-E)]x(t)\,\mathrm{d}t\right],\end{aligned}$$

or, equivalently, since

$$\begin{aligned}&\mathbb{E}\left[\int_0^\infty x^{\mathrm{T}}(t)[R+(D^{\mathrm{T}}P-E)^{\mathrm{T}}(E_\infty+E_\infty^{\mathrm{T}})^{-1}(D^{\mathrm{T}}P-E)]x(t)\,\mathrm{d}t\right]\\ &\quad=\mathbb{E}\left[\int_0^\infty x_0^{\mathrm{T}}\mathrm{e}^{A^{\mathrm{T}}t}[R+(D^{\mathrm{T}}P-E)^{\mathrm{T}}(E_\infty+E_\infty^{\mathrm{T}})^{-1}(D^{\mathrm{T}}P-E)]\mathrm{e}^{At}x_0\,\mathrm{d}t\right]\\ &\quad=\mathbb{E}[x_0^{\mathrm{T}}Px_0]=\operatorname{tr}PV,\end{aligned}$$

$\|G\|_2^2=\operatorname{tr}\hat{P}V\leq\operatorname{tr}PV$ which implies that $J(x_0)$ given by (41) provides an upper bound to the H_2 norm of $G(s)$.

Next we present a result in which $\mathscr{W}$ is assumed to be a bounded subset of L_2. Specifically, define the subset of square-integrable bounded disturbances

$$\mathscr{W}_\beta\triangleq\left\{w(\cdot)\colon\int_0^T w^{\mathrm{T}}(t)w(t)\,\mathrm{d}t\leq\beta,\ T\geq 0\right\}, \tag{54}$$

where $\beta>0$. Furthermore, let $L:\mathscr{D}\to\mathbb{R}$ be given such that $L(x)\geq 0,\ x\in\mathscr{D}$.

Theorem 3.2. *Let $\gamma>0$ and consider the nonlinear system* (7) *with performance functional* (14). *Assume there exists a* C^1 *function $V:\mathscr{D}\to\mathbb{R}$ such that*

$$V(0)=0, \tag{55}$$

$$V(x)>0,\quad x\in\mathscr{D},\ x\neq 0, \tag{56}$$

$$V'(x)f(x)<0, \quad x\in\mathscr{D},\ x\neq 0, \tag{57}$$

$$L(x)+V'(x)f(x)+\frac{\beta}{4\gamma}V'(x)J_1(x)J_1^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)=0, \quad x\in\mathscr{D}. \tag{58}$$

Then there exists a neighborhood $\mathscr{D}_0\subset\mathscr{D}$ of the origin such that if $x_0\in\mathscr{D}_0$, then $x(t)\equiv 0$ is a locally asymptotically stable solution of (7) *with $w(t)\equiv 0$. Then*

$$J(x_0)\leq\mathscr{J}(x_0)=V(x_0), \tag{59}$$

where

$$\mathscr{J}(x_0)\triangleq\int_0^\infty [L(x(t))+\Gamma(x(t))]\,\mathrm{d}t, \tag{60}$$

$$\Gamma(x)=\frac{\beta}{4\gamma}V'(x)J_1(x)J_1^{\mathrm{T}}(x)V'^{\mathrm{T}}(x), \tag{61}$$

and where $x(t)$, $t\geq 0$, solves (7) *with $w(t)\equiv 0$. Furthermore, if $x_0=0$ then the solution $x(t)$, $t\geq 0$, of* (7) *satisfies*

$$V(x(T))\leq\gamma, \quad T\geq 0,\ w(\cdot)\in\mathscr{W}_\beta. \tag{62}$$

Finally, if $\mathscr{D}=\mathbb{R}^n$, $w(t)\equiv 0$, and

$$V(x)\to\infty \quad \text{as } \|x\|\to\infty, \tag{63}$$

then the solution $x(t)=0$, $t\geq 0$, of (7) *is globally asymptotically stable.*

Proof. The proofs for local and global asymptotic stability and the performance bound (59) are identical to the proofs of local and global asymptotic stability given in Theorem 3.1 and the performance bound (20). Next, with $r(z,w)=\frac{\gamma}{\beta}w^{\mathrm{T}}w$ and $\Gamma(x)$ given by (61) it follows from Lemma 3.1 that

$$V(x(T))\leq\frac{\gamma}{\beta}\int_0^T w^{\mathrm{T}}(t)w(t)\,\mathrm{d}t, \quad T\geq 0,\ w(\cdot)\in\mathscr{W}_\beta,$$

which yields (62). □

3.2. A Riccati equation characterization for mixed $\mathrm{H}_2/\mathrm{L}_1$ performance

In this section we consider yet another linearization of the nonlinear system (2), (3) with bounded amplitude persistent L_∞ disturbances. Specifically, let $f(x)=Ax$, $J_1(x)=D$, $h(x)=Ex$, and $J_2(x)=E_\infty$ where $A\in\mathbb{R}^{n\times n}$, $D\in\mathbb{R}^{n\times d}$, $E\in\mathbb{R}^{p\times n}$, $E_\infty\in\mathbb{R}^{p\times d}$, and A is asymptotically stable so that

$$\dot{x}(t)=Ax(t)+Dw(t), \quad x(0)=0,\ t\geq 0,\ w(\cdot)\in\hat{\mathscr{W}}, \tag{64}$$

$$z(t)=Ex(t)+E_\infty w(t), \tag{65}$$

where $\hat{\mathscr{W}}$ consists of unit-peak input signals defined by

$$\hat{\mathscr{W}}\triangleq\{w(\cdot)\colon w^{\mathrm{T}}(t)w(t)\leq 1,\ t\geq 0\}. \tag{66}$$

The following result provides an upper bound to the L_1 norm (L_∞ equi-induced norm) of the convolution operator G of the linear time-invariant system (64), (65) given by

$$\|G\|_1 \triangleq \sup_{w(\cdot)\in\hat{\mathscr{W}}} \left\{ \sup_{t\geq 0} \|z(t)\| \right\},$$

where $\|\cdot\|$ denotes the Euclidean vector norm. From an input–output point of view the L_1 norm captures the worst-case amplification from input disturbance signals to output signals, where the signal size is taken to be the supremum over time of the signal's point-wise-in-time Euclidean norm.

Theorem 3.3. *Let* $\alpha>0$ *and consider the linear system* (64) *and* (65). *Then*

$$\|G\|_1 \leq \sigma_{\max}^{1/2}(EP^{-1}E^{\mathrm{T}}) + \sigma_{\max}^{1/2}(E_\infty E_\infty^{\mathrm{T}}), \tag{67}$$

where $P>0$ *satisfies*

$$0 \geq A^{\mathrm{T}}P + PA + \alpha P + \frac{1}{\alpha} PDD^{\mathrm{T}}P. \tag{68}$$

Proof. Let $T\geq 0$ and consider the shifted linear system

$$\dot{\tilde{x}}(t) = A_\alpha \tilde{x}(t) + Dv(t), \quad \tilde{x}(0)=0, \ t\geq 0, \tag{69}$$

$$z(t) = \mathrm{e}^{-\frac{\alpha}{2}(t-T)}(E\tilde{x}(t) + E_\infty v(t)), \tag{70}$$

where $A_\alpha \triangleq A + \frac{\alpha}{2}I$, $\tilde{x}(t) \triangleq \mathrm{e}^{\frac{\alpha}{2}(t-T)}x(t)$, and $v(t) \triangleq \mathrm{e}^{\frac{\alpha}{2}(t-T)}w(t)$. Note that (69), (70) are equivalent to (64), (65). Furthermore, note that if $w(\cdot)\in\hat{\mathscr{W}}$ then $v(\cdot)\in\mathscr{V}$ where $\mathscr{V} \triangleq \{v(\cdot)\colon \int_0^{\mathrm{T}} v^{\mathrm{T}}(t)v(t)\,\mathrm{d}t \leq \frac{1}{\alpha}\}$. Hence

$$\|G\|_1^2 \leq \sup_{T\geq 0} \sup_{v(\cdot)\in\mathscr{V}} \|z(T)\|^2.$$

Next, with $f(x)=A_\alpha\tilde{x}(t)$, $J_1(x)=D$, $V(x)=\tilde{x}^{\mathrm{T}}P\tilde{x}$, $\mathscr{W}=\mathscr{V}$, $\beta=\frac{1}{\alpha}$, $\gamma=1$, and $L(x)=\tilde{x}^{\mathrm{T}}R\tilde{x}$ where $R\in\mathbb{R}^{n\times n}$ is an arbitrary positive definite matrix it follows from Theorem 3.2 that if there exists $P>0$ such that

$$0 = A_\alpha^{\mathrm{T}}P + PA_\alpha + \frac{1}{\alpha}PDD^{\mathrm{T}}P + R, \tag{71}$$

then

$$\tilde{x}^{\mathrm{T}}(T)P\tilde{x}(T) \leq 1,$$

and hence for all $v(\cdot)\in\mathscr{V}$

$$\|z(T)\| = \|E\tilde{x}(T) + E_\infty w(T)\| \leq \sigma_{\max}^{1/2}(EP^{-1}E^{\mathrm{T}}) + \sigma_{\max}^{1/2}(E_\infty E_\infty^{\mathrm{T}}).$$

The result is now immediate by noting that (71) is equivalent to (68). □

Remark 3.2. Note that we can replace the Riccati inequality (68) by the Riccati equation

$$0 = A^{\mathrm{T}}P + PA + \alpha P + \frac{1}{\alpha} PDD^{\mathrm{T}}P, \tag{72}$$

in Theorem 3.3. In this case if (A,D) is controllable and A_α is asymptotically stable then there exists a positive-definite solution satisfying (72). In particular,

$$P = \left[\int_0^\infty \mathrm{e}^{A_\alpha t} DD^{\mathrm{T}} \mathrm{e}^{A_\alpha^{\mathrm{T}} t}\, \mathrm{d}t \right]^{-1}. \tag{73}$$

Remark 3.3. Letting $\mathscr{Q} = \alpha P^{-1}$ in Theorem 3.3 yields

$$\|G\|_1 \le \frac{1}{\alpha} \sigma_{\max}^{1/2}(E\mathscr{Q}E^{\mathrm{T}}) + \sigma_{\max}^{1/2}(E_\infty E_\infty^{\mathrm{T}}), \tag{74}$$

where $\mathscr{Q} > 0$ satisfies

$$0 \ge A\mathscr{Q} + \mathscr{Q}A^{\mathrm{T}} + \alpha\mathscr{Q} + DD^{\mathrm{T}}. \tag{75}$$

Furthermore, it is interesting to note that in the case where $E_\infty = 0$ the solution $\mathscr{Q}$ to (75) satisfies the bound

$$Q \le \mathscr{Q}, \tag{76}$$

where Q satisfies

$$0 = AQ + QA^{\mathrm{T}} + DD^{\mathrm{T}} \tag{77}$$

and hence

$$\|G\|_2^2 = \operatorname{tr} EQE^{\mathrm{T}} \le \operatorname{tr} E\mathscr{Q}E^{\mathrm{T}}. \tag{78}$$

Thus, (75) can be used to provide a trade-off between H_2 and mixed $\mathrm{H}_2/\mathrm{L}_1$ performance. For further details see [41].

4. Nonlinear–nonquadratic controllers for systems with bounded disturbances

In this section we consider a control problem involving a notion of optimality with respect to an *auxiliary cost* which guarantees a bound on the worst-case value of a nonlinear–nonquadratic cost functional over a prescribed set of bounded input disturbances. The optimal feedback controllers are derived as a direct consequence of Theorem 3.1 and provide a transparent generalization of the Hamilton–Jacobi–Bellman conditions for time-invariant, infinite horizon problems for addressing nonlinear feedback controllers for nonlinear systems with bounded energy disturbances that additionally minimize a nonlinear–nonquadratic cost functional. To address the optimal control problem let $\mathscr{D} \subset \mathbb{R}^n$ be an open set and let $\mathscr{C} \subset \mathbb{R}^m$, where $0 \in \mathscr{D}$ and $0 \in \mathscr{C}$.

Furthermore, let $\mathscr{W} \subset \mathbb{R}^d$ be a subset of measurable functions and let $r : \mathbb{R}^p \times \mathbb{R}^d \to \mathbb{R}$ be a given function. Next, consider the controlled system

$$\dot{x}(t) = \tilde{f}(x(t), u(t)) + J_1(x(t))w(t), \quad x(0) = x_0, \ t \geq 0, \tag{79}$$

with performance variables

$$z(t) = \tilde{h}(x(t), u(t)) + J_2(x(t))w(t), \tag{80}$$

where $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ such that $\tilde{f}(0,0) = 0$, $J_1 : \mathbb{R}^n \to \mathbb{R}^{n\times d}$, $\tilde{h} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$, $J_2 : \mathbb{R}^n \to \mathbb{R}^{p\times d}$ and the control $u(\cdot)$ is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in \mathscr{U}$ for all $t \geq 0$ where the control constraint set $\mathscr{U} \subset \mathscr{C}$ is given. We assume $0 \in \mathscr{U}$. A measurable mapping $\phi : \mathscr{D} \to \mathscr{U}$ satisfying $\phi(0) = 0$ is called a *control law*. If $u(t) = \phi(x(t))$, where $\phi(\cdot)$ is a control law and $x(t)$ satisfies (79), then $u(\cdot)$ is called a *feedback control*. Given a control law $\phi(\cdot)$ and a feedback control $u(t) = \phi(x(t))$, the *closed-loop system* has the form

$$\dot{x}(t) = \tilde{f}(x(t), \phi(x(t))) + J_1(x(t))w(t), \quad x(0) = x_0, \ t \geq 0, \tag{81}$$

$$z(t) = \tilde{h}(x(t), \phi(x(t))) + J_2(x(t))w(t). \tag{82}$$

We assume that the mapping $\phi : \mathscr{D} \to \mathscr{U}$ satisfies sufficient regularity conditions such that the resulting closed-loop system (81) has a unique solution forward in time.

Next we present an extension of Theorem 4.1 of [31] for characterizing feedback controllers that guarantee stability, minimize an auxiliary performance functional, and guarantee that the input–output map of the closed-loop system is dissipative, nonexpansive, and passive for bounded input disturbances. For the statement of this result let $\tilde{L} : \mathscr{D} \times \mathscr{U} \to \mathbb{R}$ and define the set of asymptotically stabilizing controllers for the nonlinear system with $w(t) \equiv 0$ by

$$\mathscr{S}(x_0) \triangleq \{u(\cdot)\colon\ u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (79) satisfies } x(t) \to 0 \text{ as } t \to \infty \text{ with } w(t) \equiv 0\}.$$

Theorem 4.1. *Consider the controlled system* (79), (80) *with performance functional*

$$\tilde{J}(x_0, u(\cdot)) \triangleq \int_0^\infty \tilde{L}(x(t), u(t))\,\mathrm{d}t, \tag{83}$$

where $u(\cdot)$ is an admissible control. Assume there exist functions $V : \mathscr{D} \to \mathbb{R}$, $\tilde{\Gamma} : \mathscr{D} \times \mathscr{U} \to \mathbb{R}$, and control law $\phi : \mathscr{D} \to \mathscr{U}$ where $V(\cdot)$ is a C^1 function such that

$$V(0) = 0, \tag{84}$$

$$V(x) > 0, \quad x \in \mathscr{D}, \ x \neq 0, \tag{85}$$

$$\phi(0) = 0, \tag{86}$$

$$V'(x)\tilde{f}(x, \phi(x)) < 0, \quad x \in \mathscr{D}, \ x \neq 0, \tag{87}$$

$$V'(x)J_1(x)w \leq r(z, w) + \tilde{L}(x, \phi(x)) + \tilde{\Gamma}(x, \phi(x)), \quad x \in \mathscr{D}, \ w \in \mathscr{W}, \tag{88}$$

$$H(x,\phi(x))=0, \quad x\in\mathscr{D}, \tag{89}$$

$$H(x,u)\geq 0, \quad x\in\mathscr{D},\ u\in\mathscr{U}, \tag{90}$$

where

$$H(x,u)\triangleq V'(x)\tilde{f}(x,u)+\tilde{L}(x,u)+\tilde{\Gamma}(x,u). \tag{91}$$

Then, with the feedback control $u(\cdot)=\phi(x(\cdot))$, *there exists a neighborhood* $\mathscr{D}_0\subset\mathscr{D}$ *of the origin such that if* $x_0\in\mathscr{D}_0$ *and* $w(t)\equiv 0$, *the solution* $x(t)\equiv 0$ *of the closed-loop system* (81) *is locally asymptotically stable. If, in addition,* $\Gamma(x,\phi(x))\geq 0$, $x\in\mathscr{D}$, *then*

$$\tilde{J}(x_0,\phi(x(\cdot)))\leq\tilde{\mathscr{J}}(x_0,\phi(x(\cdot)))=V(x_0), \tag{92}$$

where

$$\tilde{\mathscr{J}}(x_0,u(\cdot))\triangleq\int_0^\infty[\tilde{L}(x(t),u(t))+\tilde{\Gamma}(x(t),u(t))]\,\mathrm{d}t \tag{93}$$

and where $u(\cdot)$ *is admissible and* $x(t)$, $t\geq 0$, *solves* (79) *with* $w(t)\equiv 0$. *In addition, if* $x_0\in\mathscr{D}_0$ *then the feedback control* $u(\cdot)=\phi(x(\cdot))$ *minimizes* $\tilde{\mathscr{J}}(x_0,u(\cdot))$ *in the sense that*

$$\tilde{\mathscr{J}}(x_0,\phi(x(\cdot)))=\min_{u(\cdot)\in\mathscr{S}(x_0)}\tilde{\mathscr{J}}(x_0,u(\cdot)). \tag{94}$$

Furthermore, the solution $x(t)$, $t\geq 0$, *of* (81) *satisfies the dissipativity constraint*

$$\int_0^{\mathrm{T}} r(z(t),w(t))\,\mathrm{d}t+V(x_0)\geq 0, \quad T\geq 0,\ w(\cdot)\in\mathscr{W}. \tag{95}$$

Finally, if $\mathscr{D}=\mathbb{R}^n$, $w(t)\equiv 0$, *and*

$$V(x)\to\infty \quad \text{as } \|x\|\to\infty, \tag{96}$$

then the solution $x(t)=0$, $t\geq 0$, *of the closed-loop system* (81) *is globally asymptotically stable.*

Proof. Local and global asymptotic stability is a direct consequence of (84)–(87) by applying Theorem 3.1 to the closed-loop system (81). Furthermore, using (89), the performance bound (92) is a restatement of (20) as applied to the closed-loop system. Next, let $u(\cdot)\in\mathscr{S}(x_0)$ and let $x(t)$, $t\geq 0$, be the solution of (79) with $w(t)\equiv 0$. Then it follows that

$$0=-\dot{V}(x(t))+V'(x(t))\tilde{f}(x(t),u(t)).$$

Hence

$$\begin{aligned}
&\tilde{L}(x(t),u(t))+\tilde{\Gamma}(x(t),u(t))\\
&\quad=-\dot{V}(x(t))+\tilde{L}(x(t),u(t))+V'(x(t))\tilde{f}(x(t),u(t))+\tilde{\Gamma}(x(t),u(t))\\
&\quad=-\dot{V}(x(t))+H(x(t),u(t)).
\end{aligned}$$

Now using (90) and (92) and the fact that $u(\cdot)\in\mathscr{S}(x_0)$, it follows that

$$\begin{aligned}
\tilde{\mathscr{J}}(x_0,u(\cdot))&=\int_0^\infty[-\dot{V}(x(t))+H(x(t),u(t))]\,\mathrm{d}t\\
&=-\lim_{t\to\infty}V(x(t))+V(x_0)+\int_0^\infty H(x(t),u(t))\,\mathrm{d}t\\
&=V(x_0)+\int_0^\infty H(x(t),u(t))\,\mathrm{d}t\\
&\geq V(x_0)\\
&=\tilde{\mathscr{J}}(x_0,\phi(x(\cdot))),
\end{aligned}$$

which yields (94). Finally, using (88), condition (95) is a restatement of (22) as applied to the closed-loop system. □

Remark 4.1. Note that (89) is the steady state (time-independent) algebraic Hamilton–Jacobi–Bellman equation for the nonlinear system $\tilde{f}(\cdot,\cdot)$ with auxiliary cost $\tilde{\mathscr{J}}(x_0,u(\cdot))$.

Remark 4.2. Theorem 4.1 guarantees optimality with respect to the set of admissible stabilizing controllers $\mathscr{S}(x_0)$. However, it is important to note that an explicit characterization of $\mathscr{S}(x_0)$ is not required. Furthermore, the optimal stabilizing control law $u=\phi(x)$ is a feedback controller and its optimality is independent of the initial condition x_0.

Next, we specialize Theorem 4.1 to linear systems with bounded energy disturbances and provide connections to the mixed-norm $\mathrm{H}_2/\mathrm{H}_\infty$ and mixed H_2/positivity frameworks developed in [12] and [33, 36], respectively. Specifically, we consider the case in which $\tilde{f}(x,u)=Ax+Bu$, $J_1(x)=D$, $h(x,u)=E_1x+E_2u$, and $J_2(x)=E_\infty$ where $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $D\in\mathbb{R}^{n\times d}$, $E_1\in\mathbb{R}^{p\times n}$, $E_2\in\mathbb{R}^{p\times m}$, and $E_\infty\in\mathbb{R}^{p\times d}$.

First, we consider the mixed-norm $\mathrm{H}_2/\mathrm{H}_\infty$ case where $\mathscr{W}=\mathrm{L}_2$ and $r(z,w)=\gamma^2w^{\mathrm{T}}w-z^{\mathrm{T}}z$ where $\gamma>0$ is given. For the following result define $R_1\triangleq E_1^{\mathrm{T}}E_1>0$, $R_2\triangleq E_2^{\mathrm{T}}E_2>0$, $S\triangleq BR_2^{-1}B^{\mathrm{T}}$, and assume $E_\infty=0$ and $E_1^{\mathrm{T}}E_2=0$.

Corollary 4.1. *Consider the linear system*

$$\dot{x}(t)=Ax(t)+Bu(t)+Dw(t),\quad x(0)=x_0,\ t\geq 0,\ w(\cdot)\in\mathrm{L}_2, \tag{97}$$

$$z(t)=E_1x(t)+E_2u(t), \tag{98}$$

with performance functional

$$\tilde{J}(x_0, u(\cdot)) = \int_0^\infty [x^{\mathrm{T}}(t)R_1x(t) + u^{\mathrm{T}}(t)R_2u(t)]\,\mathrm{d}t, \tag{99}$$

where $u(\cdot)$ *is admissible. Assume there exists* $P \in \mathbb{P}^{n\times n}$ *such that*

$$0 = A^{\mathrm{T}}P + PA + R_1 + \gamma^{-2}PDD^{\mathrm{T}}P - PSP. \tag{100}$$

Then with $w(t) \equiv 0$, (97) *is globally asymptotically stable for all* $x_0 \in \mathbb{R}^n$ *with the feedback control law* $u = \phi(x) = -R_2^{-1}B^{\mathrm{T}}Px$, *and*

$$\tilde{J}(x_0, \phi(x(\cdot))) \le \tilde{\mathscr{J}}(x_0, \phi(x(\cdot))) = x_0^{\mathrm{T}}Px_0, \tag{101}$$

where

$$\tilde{\mathscr{J}}(x_0, u(\cdot)) = \int_0^\infty [x^{\mathrm{T}}(t)(R_1 + \gamma^{-2}PDD^{\mathrm{T}}P)x(t) + u^{\mathrm{T}}(t)R_2u(t)]\,\mathrm{d}t, \tag{102}$$

and where $u(\cdot)$ *is admissible and* $x(t)$, $t \ge 0$, *solves* (97) *with* $w(t) \equiv 0$. *Furthermore,*

$$\tilde{\mathscr{J}}(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathscr{S}(x_0)} \tilde{\mathscr{J}}(x_0, u(\cdot)), \tag{103}$$

where $\mathscr{S}(x_0)$ *is the set of asymptotically stabilizing controllers for the system* (97) *with* $w(t) \equiv 0$ *and* $x_0 \in \mathbb{R}^n$. *Furthermore, if* $x_0 = 0$ *then with* $u = \phi(x)$, *the solution* $x(t)$, $t \ge 0$, *of* (97) *satisfies the nonexpansivity constraint*

$$\int_0^T z(t)^{\mathrm{T}}z(t)\,\mathrm{d}t \le \gamma^2 \int_0^T w(t)^{\mathrm{T}}w(t)\,\mathrm{d}t, \quad T \ge 0,\ w(\cdot) \in \mathrm{L}_2, \tag{104}$$

or, equivalently, $\|\tilde{G}(s)\|_\infty \le \gamma$ *where* $\tilde{G}(s) \sim \left[\begin{array}{c|c} A + BK & D \\ \hline E_1 + E_2K & 0 \end{array}\right]$ *and* $K \triangleq -R_2^{-1}B^{\mathrm{T}}P$.

Proof. The result is a direct consequence of Theorem 4.1 with $\tilde{f}(x,u) = Ax + Bu$, $J_1(x) = D$, $\tilde{L}(x,u) = x^{\mathrm{T}}R_1x + u^{\mathrm{T}}R_2u$, $V(x) = x^{\mathrm{T}}Px$, $\tilde{\Gamma}(x,u) = \gamma^{-2}x^{\mathrm{T}}PDD^{\mathrm{T}}Px$, $\mathscr{D} = \mathbb{R}^n$, and $\mathscr{U} = \mathbb{R}^m$. Specifically, conditions (84)–(87) are trivially satisfied. Now forming x^{T}(100) x it follows that, after some algebraic manipulations, $V'(x)J_1(x)w \le r(z,w) + \tilde{L}(x,\phi(x)) + \tilde{\Gamma}(x,\phi(x))$, for all $x \in \mathscr{D}$ and $w(\cdot) \in \mathscr{W} = \mathrm{L}_2$. Furthermore, it follows from (100) that $H(x,\phi(x)) = 0$ and $H(x,u) = H(x,u) - H(x,\phi(x)) = [u - \phi(x)]^{\mathrm{T}}R_2[u - \phi(x)] \ge 0$ so that all conditions of Theorem 4.1 are satisfied. Finally, since $V(x)$ is radially unbounded (97), with $u(t) = \phi(x(t)) = -R_2^{-1}B^{\mathrm{T}}Px(t)$, is globally asymptotically stable. □

Remark 4.3. Note that in the case where $\mathscr{U} = \mathbb{R}^m$ our feedback control $u = \phi(x)$ is globally optimal since it minimizes $H(x,u)$ and satisfies (89). Specifically, setting

$$\frac{\partial}{\partial u}H(x,u) = 0, \tag{105}$$

yields the feedback control

$$\phi(x) = -R_2^{-1}B^{\mathrm{T}}Px. \tag{106}$$

Now, since

$$\frac{\partial^2}{\partial u^2}H(x,u)=R_2>0, \tag{107}$$

it follows that for all $x\in\mathbb{R}^n$ the feedback control given by (106) minimizes $H(x,u)$. In particular, the optimal feedback control law $\phi(x)$ in Corollary 4.1 is derived using the properties of $H(x,u)$ as defined in Theorem 4.1. Specifically, since $H(x,u)=x^{\mathrm{T}}(A^{\mathrm{T}}P+PA+R_1+\gamma^{-2}PDD^{\mathrm{T}}P)x+u^{\mathrm{T}}R_2u+2x^{\mathrm{T}}PBu$ it follows that $\partial^2H/\partial u^2>0$. Now, $\partial H/\partial u=2R_2u+2B^{\mathrm{T}}Px=0$ gives the unique global minimum of $H(x,u)$. Hence, since $\phi(x)$ minimizes $H(x,u)$ it follows that $\phi(x)$ satisfies $\partial H/\partial u=0$ or, equivalently, $R_2\phi(x)+B^{\mathrm{T}}Px=0$ so that $\phi(x)$ is given by (106). Similar remarks hold for the nonlinear controllers developed in Corollary 4.2 and Section 5.

Next, we specialize Theorem 4.1 to provide connections to the mixed H_2/positivity framework developed in [33, 36]. Specifically, we consider the case where $p=d$, $\mathscr{W}=\mathrm{L}_2$, and $r(z,w)=2w^{\mathrm{T}}z$. For the following result define $R_0\triangleq(E_\infty+E_\infty^{\mathrm{T}})^{-1}$, $R_{2\mathrm{s}}\triangleq R_2+E_2^{\mathrm{T}}R_0E_2$, $R_{1\mathrm{s}}\triangleq E_1^{\mathrm{T}}(I+R_0)E_1-E_1^{\mathrm{T}}R_0E_2R_{2\mathrm{s}}^{-1}E_2^{\mathrm{T}}R_0E_1$, $B_{\mathrm{s}}\triangleq B-DR_0E_2$, $A_{\mathrm{s}}\triangleq A-(B_{\mathrm{s}}R_{2\mathrm{s}}^{-1}E_2^{\mathrm{T}}+D)R_0E_1$, and $S_{\mathrm{s}}\triangleq B_{\mathrm{s}}R_{2\mathrm{s}}^{-1}B_{\mathrm{s}}^{\mathrm{T}}$. Furthermore, assume that $E_1^{\mathrm{T}}E_1>0$ and $E_2^{\mathrm{T}}E_2>0$. Note that using Schur complements it can be shown that $R_{1\mathrm{s}}>0$.

Corollary 4.2. *Consider the linear system* (97) *with performance variables*

$$z(t)=E_1x(t)+E_2u(t)+E_\infty w(t), \tag{108}$$

and performance functional (99). *Assume there exists* $P\in\mathbb{P}^{n\times n}$ *such that*

$$0=A_{\mathrm{s}}^{\mathrm{T}}P+PA_{\mathrm{s}}+R_{1\mathrm{s}}+PDR_0D^{\mathrm{T}}P-PS_{\mathrm{s}}P. \tag{109}$$

Then with $w(t)\equiv0$, (97) *is globally asymptotically stable for all* $x_0\in\mathbb{R}^n$ *with the feedback control law* $u=\phi(x)=-R_{2\mathrm{s}}^{-1}(B_{\mathrm{s}}^{\mathrm{T}}P+E_2^{\mathrm{T}}R_0E_1)x$, *and*

$$\tilde{J}(x_0,\phi(x(\cdot)))\le\tilde{\mathscr{J}}(x_0,\phi(x(\cdot)))=x_0^{\mathrm{T}}Px_0, \tag{110}$$

where

$$\begin{aligned}\tilde{\mathscr{J}}(x_0,u(\cdot))=&\int_0^\infty[x^{\mathrm{T}}(t)(R_1+(D^{\mathrm{T}}P-E_1)^{\mathrm{T}}R_0(D^{\mathrm{T}}P-E_1))x(t)+u^{\mathrm{T}}(t)R_{2\mathrm{s}}u(t)\\&-2x^{\mathrm{T}}(t)(D^{\mathrm{T}}P-E_1)^{\mathrm{T}}R_0E_2u(t)]\,\mathrm{d}t\end{aligned} \tag{111}$$

and where $u(\cdot)$ *is admissible and* $x(t)$, $t\ge0$, *solves* (97) *with* $w(t)\equiv0$. *Furthermore,*

$$\tilde{\mathscr{J}}(x_0,\phi(x(\cdot)))=\min_{u(\cdot)\in\mathscr{S}(x_0)}\tilde{\mathscr{J}}(x_0,u(\cdot)), \tag{112}$$

where $\mathscr{S}(x_0)$ *is the set of asymptotically stabilizing controllers for the system* (97) *with* $w(t)\equiv0$ *and* $x_0\in\mathbb{R}^n$. *Furthermore, if* $x_0=0$ *then with* $u=\phi(x)$, *the solution* $x(t)$, $t\ge0$, *of* (97) *satisfies the passivity constraint*

$$\int_0^T w^{\mathrm{T}}(t)z(t)\ge0,\quad T\ge0,\ w(\cdot)\in\mathrm{L}_2, \tag{113}$$

or, equivalently, $\tilde{G}_\infty(s)+\tilde{G}^*_\infty(s)\geq 0$, $\mathrm{Re}[s]>0$, *where* $\tilde{G}_\infty(s)\sim\left[\begin{array}{c|c} A+BK & D \\ \hline E_1+E_2K & E_\infty\end{array}\right]$ *and* $K=-R_{2s}^{-1}(B_s^{\mathrm{T}}P+E_2^{\mathrm{T}}R_0E_1)$.

Proof. The result is a direct consequence of Theorem 4.1 with $\tilde{f}(x,u)=Ax+Bu$, $J_1(x)=D$, $\tilde{L}(x,u)=x^{\mathrm{T}}R_1x+u^{\mathrm{T}}R_2u$, $V(x)=x^{\mathrm{T}}Px$, $\tilde{\Gamma}(x,u)=[(D^{\mathrm{T}}P-E_1)x-E_2u]^{\mathrm{T}}R_0[(D^{\mathrm{T}}\cdot P-E_1)x-E_2u]$, $\mathscr{D}=\mathbb{R}^n$, and $\mathscr{U}=\mathbb{R}^m$. Specifically, conditions (84)–(87) are trivially satisfied. Now forming $x^{\mathrm{T}}(109)x$ it follows that, after some algebraic manipulations, $V'(x)J_1(x)w\leq r(z,w)+\tilde{L}(x,\phi(x))+\tilde{\Gamma}(x,\phi(x))$, for all $x\in\mathscr{D}$ and $w(\cdot)\in\mathscr{W}=\mathrm{L}_2$. Furthermore, it follows from (109) that $H(x,\phi(x))=0$ and $H(x,u)=H(x,u)-H(x,\phi(x))=[u-\phi(x)]^{\mathrm{T}}R_{2s}[u-\phi(x)]\geq 0$ so that all conditions of Theorem 4.1 are satisfied. Finally, since $V(x)$ is radially unbounded (97), with $u(t)=\phi(x(t))=-R_{2s}^{-1}(B_s^{\mathrm{T}}P+E_2^{\mathrm{T}}R_0E_1)x(t)$, is globally asymptotically stable. □

5. Nonlinear controllers with nonlinear–nonquadratic performance criteria

In this section we specialize the results of Section 4 to linear systems controlled by nonlinear controllers that minimize a nonlinear–nonquadratic cost functional. Specifically, we consider the linear system (97) controlled by nonlinear controllers. First, we consider the case in which $r(z,w)=\gamma^2w^{\mathrm{T}}w-z^{\mathrm{T}}z$. Recall the definitions of S, R_1, and R_2 and let $\ell:\mathbb{R}^n\to\mathbb{R}$ be a multilinear function such that $\ell(x)\geq 0$, $x\in\mathbb{R}^n$. Furthermore, assume that $\gamma^{-2}DD^{\mathrm{T}}\leq S$ where $\gamma>0$ is given.

Proposition 5.1. *Consider the system* (97), (98) *and assume there exists a matrix* $P\in\mathbb{P}^{n\times n}$ *and a function* $p:\mathbb{R}^n\to\mathbb{R}$ *such that* $p(x)\geq 0$, $x\in\mathbb{R}^n$, *and*

$$0=A^{\mathrm{T}}P+PA+R_1+\gamma^{-2}PDD^{\mathrm{T}}P-PSP \tag{114}$$

and

$$0=p'(x)[A-(S-\gamma^{-2}DD^{\mathrm{T}})P]x+\ell(x). \tag{115}$$

Furthermore, let

$$\tilde{L}(x,u)=x^{\mathrm{T}}R_1x+\ell(x)+\tfrac{1}{4}p'(x)(S-\gamma^{-2}DD^{\mathrm{T}})p'^{\mathrm{T}}(x)+u^{\mathrm{T}}R_2u, \tag{116}$$

and

$$\tilde{\Gamma}(x,u)=\gamma^{-2}[Px+\tfrac{1}{2}p'^{\mathrm{T}}(x)]^{\mathrm{T}}DD^{\mathrm{T}}[Px+\tfrac{1}{2}p'^{\mathrm{T}}(x)]. \tag{117}$$

Then the system (97) *with* $w(t)\equiv 0$ *and performance functional*

$$\tilde{J}(x_0,u(\cdot))=\int_0^\infty \tilde{L}(x(t),u(t))\,\mathrm{d}t, \tag{118}$$

is globally asymptotically stable for all $x_0\in\mathbb{R}^n$ *with the feedback control law* $u=\phi(x)=-R_2^{-1}B^{\mathrm{T}}(Px+\frac{1}{2}p'^{\mathrm{T}}(x))$, *and*

$$\tilde{J}(x_0,\phi(x(\cdot)))\leq\tilde{\mathscr{J}}(x_0,\phi(x(\cdot)))=x_0^{\mathrm{T}}Px_0+p(x_0), \tag{119}$$

where

$$\tilde{\mathscr{J}}(x_0, u(\cdot)) = \int_0^\infty [\tilde{L}(x(t), u(t)) + \tilde{\Gamma}(x(t), u(t))]\,\mathrm{d}t \tag{120}$$

and where $u(\cdot)$ *is admissible and* $x(t)$, $t \geq 0$, *solves* (97) *with* $w(t) \equiv 0$. *Furthermore,*

$$\tilde{\mathscr{J}}(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathscr{S}(x_0)} \tilde{\mathscr{J}}(x_0, u(\cdot)), \tag{121}$$

where $\mathscr{S}(x_0)$ *is the set of asymptotically stabilizing controllers for the system* (97) *with* $w(t) \equiv 0$ *and* $x_0 \in \mathbb{R}^n$. *Furthermore, with* $u = \phi(x)$, *the solution* $x(t)$, $t \geq 0$, *of* (97) *satisfies the nonexpansivity constraint*

$$\int_0^T z^{\mathrm{T}}(t) z(t)\,\mathrm{d}t \leq \gamma^2 \int_0^T w(t)^{\mathrm{T}} w(t)\,\mathrm{d}t + V(x_0), \quad T \geq 0,\ w(\cdot) \in \mathrm{L}_2. \tag{122}$$

Proof. The result is a direct consequence of Theorem 4.1 with $\tilde{f}(x,u) = Ax + Bu$, $J_1(x) = D$, $\tilde{L}(x,u) = x^{\mathrm{T}} R_1 x + \frac{1}{2} p'(x)(S - \gamma^{-2} DD^{\mathrm{T}}) p'^{\mathrm{T}}(x) + \ell(x) + u^{\mathrm{T}} R_2 u$, $V(x) = x^{\mathrm{T}} P x + p(x)$, $\mathscr{D} = \mathbb{R}^n$, and $\mathscr{U} = \mathbb{R}^m$. Specifically, conditions (84)–(87) are trivially satisfied. Now forming $x^{\mathrm{T}}(114)x + (115)$ it follows that, after some algebraic manipulations, $V'(x) J_1(x) w \leq r(z,w) + \tilde{L}(x, \phi(x)) + \tilde{\Gamma}(x, \phi(x))$, for all $x \in \mathscr{D}$ and $w(\cdot) \in \mathscr{W} = \mathrm{L}_2$. Furthermore, it follows from (114) and (115) that $H(x, \phi(x)) = 0$ and $H(x,u) = H(x,u) - H(x, \phi(x)) = [u - \phi(x)]^{\mathrm{T}} R_2 [u - \phi(x)] \geq 0$ so that all conditions of Theorem 4.1 are satisfied. Finally, since $V(x)$ is radially unbounded (97), with $u(t) = \phi(x(t)) = -R_2^{-1}(B^{\mathrm{T}} P x(t) + \frac{1}{2} p'^{\mathrm{T}}(x(t)))$, is globally asymptotically stable. □

Remark 5.1. Since $A - (S - \gamma^{-2} DD^{\mathrm{T}})P$ is Hurwitz if $\ell(x)$ is a nonnegative multilinear function then it follows from Lemma 2.1 that there exists a nonnegative $p(x)$ such that (115) is satisfied. Note that Proposition 5.1 generalizes the classic results of Bass and Webber [37] to optimal control of nonlinear systems with bounded energy disturbances.

Remark 5.2. As discussed in [31,37,38] the performance functional (116) is somewhat contrived in the sense that it cannot be arbitrarily specified. However, this performance functional does weigh the state variables by arbitrary even powers. Furthermore, it is interesting to note that (116) has the form

$$\begin{aligned} \tilde{J}(x_0, u(\cdot)) = \int_0^\infty & [x^{\mathrm{T}} R_1 x + \ell(x) + u^{\mathrm{T}} R_2 u - \tfrac{1}{4}\gamma^{-2} p'(x) DD^{\mathrm{T}} p'^{\mathrm{T}}(x) \\ & + \phi_{\mathrm{NL}}^{\mathrm{T}}(x) R_2 \phi_{\mathrm{NL}}(x)]\mathrm{d}t, \end{aligned}$$

where $\phi_{\mathrm{NL}}(x)$ is the nonlinear part of the optimal feedback control

$$\phi(x) = \phi_{\mathrm{L}}(x) + \phi_{\mathrm{NL}}(x),$$

where $\phi_{\mathrm{L}}(x) = -R_2^{-1}B^{\mathrm{T}}Px$ and $\phi_{\mathrm{NL}}(x) = -\frac{1}{2}R_2^{-1}B^{\mathrm{T}}p'^{\mathrm{T}}(x)$.

It is interesting to note that if $p(x)$ is a *polynomial function* of the form $p(x) = \sum_{k=2}^{r}\frac{1}{k}(x^{\mathrm{T}}M_kx)^k$ then it follows from (115) that $\ell(x) = \sum_{k=2}^{r}(x^{\mathrm{T}}M_kx)^{k-1}x^{\mathrm{T}}\hat{R}_kx$, where $M_k, \hat{R}_k \in \mathbb{N}^{n\times n}$, $k = 2,\ldots,r$, and M_k satisfies

$$0 = [A - (S - \gamma^{-2}DD^{\mathrm{T}}P)]^{\mathrm{T}}M_k + M_k[A - (S - \gamma^{-2}DD^{\mathrm{T}}P)] + \hat{R}_k, \quad k = 2,\ldots,r, \tag{123}$$

where P satisfies (114). In this case the optimal control law $\phi(x)$ is given by

$$\phi(x) = -R_2^{-1}B^{\mathrm{T}}\left(Px + \sum_{k=2}^{r}(x^{\mathrm{T}}M_kx)^{k-1}M_kx\right),$$

the corresponding Lyapunov function guaranteeing closed-loop stability is given by

$$V(x) = x^{\mathrm{T}}Px + \sum_{k=2}^{r}\frac{1}{k}(x^{\mathrm{T}}M_kx)^k,$$

and $\tilde{L}(x,u)$ given by (116) becomes

$$\begin{aligned}\tilde{L}(x,u) = {} & x^{\mathrm{T}}\left(R_1 + \sum_{k=2}^{r}(x^{\mathrm{T}}M_kx)^{k-1}\hat{R}_k\right)x + \left(\sum_{k=2}^{r}(x^{\mathrm{T}}M_kx)^{k-1}M_kx\right)^{\mathrm{T}} \\ & \times(S - \gamma^{-2}DD^{\mathrm{T}})\left(\sum_{k=2}^{r}(x^{\mathrm{T}}M_kx)^{k-1}M_kx\right) + u^{\mathrm{T}}R_2u.\end{aligned}$$

Furthermore, if $\hat{R}_k = \hat{R}_2$, $k = 3,\ldots,r$, then $M_k = M_2$, $k = 3,\ldots,r$, satisfies (123). In this case we only require the solution of one modified Riccati equation in (123).

Remark 5.3. Proposition 5.1 generalizes the deterministic version of the stochastic nonlinear–nonquadratic optimal control problem considered in [38] to the disturbance rejection setting. Furthermore, unlike the results of [38], this result is not limited to sixth order cost functionals and cubic nonlinear controllers since it addresses a polyquadratic nonlinear performance criterion.

Next, we consider the linear system (97) controlled by nonlinear controllers where $r(z,w) = 2z^{\mathrm{T}}w$. For the statement of the next result recall the definitions of R_0, S_{s}, $R_{1\mathrm{s}}$, $R_{2\mathrm{s}}$, A_{s}, B_{s} and let $\ell:\mathbb{R}^n \to \mathbb{R}$ be a multilinear function such that $\ell(x) \geq 0$, $x \in \mathbb{R}^n$. Furthermore, assume that $DR_0D^{\mathrm{T}} \leq S_{\mathrm{s}}$ and $p = d$.

Proposition 5.2. *Consider the system* (97), (108) *and assume there exists a matrix* $P \in \mathbb{P}^{n\times n}$ *and a function* $p:\mathbb{R}^n \to \mathbb{R}$ *such that* $p(x) \geq 0$, $x \in \mathbb{R}^n$, *and*

$$0 = A_{\mathrm{s}}^{\mathrm{T}}P + PA_{\mathrm{s}} + R_{1\mathrm{s}} + PDR_0D^{\mathrm{T}}P - PS_{\mathrm{s}}P \tag{124}$$

and

$$0 = p'(x)[A_{\mathrm{s}} - (S_{\mathrm{s}} - DR_0D^{\mathrm{T}})P]\,x + \ell(x). \tag{125}$$

Furthermore, let

$$\tilde{L}(x,u) = x^{\mathrm{T}}R_1x + \ell(x) + \tfrac{1}{4}p'(x)(S_{\mathrm{s}} - DR_0D^{\mathrm{T}})p'^{\mathrm{T}}(x) + u^{\mathrm{T}}R_2u, \tag{126}$$

and

$$\tilde{\Gamma}(x,u) = [D^{\mathrm{T}}(Px + \tfrac{1}{2}p'^{\mathrm{T}}(x)) - E_1x - E_2u]^{\mathrm{T}}R_0[D^{\mathrm{T}}(Px + \tfrac{1}{2}p'^{\mathrm{T}}(x)) - E_1x - E_2u]. \tag{127}$$

Then the system (97) *with* $w(t) \equiv 0$ *and performance functional*

$$\tilde{J}(x_0,u(\cdot)) = \int_0^\infty \tilde{L}(x(t),u(t))\,\mathrm{d}t, \tag{128}$$

is globally asymptotically stable for all $x_0 \in \mathbb{R}^n$ *with the feedback control law* $u = \phi(x) = -R_{2\mathrm{s}}^{-1}(B_{\mathrm{s}}^{\mathrm{T}}(Px + \frac{1}{2}p'^{\mathrm{T}}(x)) + E_2^{\mathrm{T}}R_0E_1x)$, *and*

$$\tilde{J}(x_0,\phi(x(\cdot))) \leq \tilde{\mathscr{J}}(x_0,\phi(x(\cdot))) = x_0^{\mathrm{T}}Px_0 + p(x_0), \tag{129}$$

where

$$\tilde{\mathscr{J}}(x_0,u(\cdot)) = \int_0^\infty [\tilde{L}(x(t),u(t)) + \tilde{\Gamma}(x(t),u(t))]\,\mathrm{d}t \tag{130}$$

and where $u(\cdot)$ *is admissible and* $x(t)$, $t \geq 0$, *solves* (97) *with* $w(t) \equiv 0$. *Furthermore,*

$$\tilde{\mathscr{J}}(x_0,\phi(x(\cdot))) = \min_{u(\cdot)\in\mathscr{S}(x_0)} \tilde{\mathscr{J}}(x_0,u(\cdot)), \tag{131}$$

where $\mathscr{S}(x_0)$ *is the set of asymptotically stabilizing controllers for the system* (97) *with* $w(t) \equiv 0$ *and* $x_0 \in \mathbb{R}^n$. *Furthermore, with* $u = \phi(x)$, *the solution* $x(t)$, $t \geq 0$, *of* (97) *satisfies the passivity constraint*

$$\int_0^T 2z^{\mathrm{T}}(t)w(t)\,\mathrm{d}t + V(x_0) \geq 0, \quad T \geq 0,\ w(\cdot) \in \mathrm{L}_2. \tag{132}$$

Proof. The result is a direct consequence of Theorem 4.1 with $\tilde{f}(x,u) = Ax + Bu$, $J_1(x) = D$, $\tilde{L}(x,u) = x^{\mathrm{T}}R_1x + \frac{1}{2}p'(x)(S_{\mathrm{s}} - DR_0D^{\mathrm{T}})p'^{\mathrm{T}}(x) + \ell(x) + u^{\mathrm{T}}R_2u$, $V(x) = x^{\mathrm{T}}Px + p(x)$, $\mathscr{D} = \mathbb{R}^n$, and $\mathscr{U} = \mathbb{R}^m$. Specifically, conditions (84)–(87) are trivially satisfied. Now forming $x^{\mathrm{T}}(124)x +$ (125) it follows that, after some algebraic manipulations, $V'(x)J_1(x)w \leq r(z,w) + \tilde{L}(x,\phi(x)) + \tilde{\Gamma}(x,\phi(x))$, for all $x \in \mathscr{D}$ and $w(\cdot) \in \mathscr{W} = \mathrm{L}_2$. Furthermore, it follows from (124) and (125) that $H(x,\phi(x)) = 0$ and $H(x,u) = H(x,u)$

$-H(x,\phi(x)) = [u-\phi(x)]^{\mathrm{T}} R_{2\mathrm{s}}[u-\phi(x)] \geq 0$ so that all conditions of Theorem 4.1 are satisfied. Finally, since $V(x)$ is radially unbounded (97), with $u(t) = \phi(x(t)) = -R_{2\mathrm{s}}^{-1}(B_{\mathrm{s}}^{\mathrm{T}}(Px(t) + \frac{1}{2}p'^{\mathrm{T}}(x(t))) + E_2^{\mathrm{T}} R_0 E_1 x(t))$, is globally asymptotically stable. □

Remark 5.4. Note that since $A_{\mathrm{s}} - (S_{\mathrm{s}} - DR_0D^{\mathrm{T}})P$ is Hurwitz if $\ell(x)$ is a nonnegative multilinear function then it follows from Lemma 2.1 that there exists a nonnegative $p(x)$ such that (125) is satisfied.

Finally, if $p(x)$ is a polynomial function of the form $\sum_{k=2}^{r} \frac{1}{k}(x^{\mathrm{T}}M_k x)^k$ then it follows from (125) that $\ell(x) = \sum_{k=2}^{r}(x^{\mathrm{T}}M_k x)^{k-1}x^{\mathrm{T}}\hat{R}_k x$, where $M_k, \hat{R}_k \in \mathbb{N}^{n\times n}$, $k = 2,\ldots,r$, and M_k satisfies

$$0 = [A_{\mathrm{s}} - (S_{\mathrm{s}} - DR_0D^{\mathrm{T}}P)]^{\mathrm{T}}M_k + M_k[A_{\mathrm{s}} - (S_{\mathrm{s}} - DR_0D^{\mathrm{T}}P)] + \hat{R}_k, \quad k = 2,\ldots,r, \tag{133}$$

and where P satisfies (124). In this case the optimal control law $\phi(x)$ is given by

$$\phi(x) = -R_{2\mathrm{s}}^{-1}\left[\left(B_{\mathrm{s}}^{\mathrm{T}}\left(Px + \sum_{k=2}^{r}(x^{\mathrm{T}}M_k x)^{k-1}M_k x\right)\right) + E_2^{\mathrm{T}}R_0E_1x\right],$$

the corresponding Lyapunov function guaranteeing closed-loop stability is given by

$$V(x) = x^{\mathrm{T}}Px + \sum_{k=2}^{r}\frac{1}{k}(x^{\mathrm{T}}M_k x)^k,$$

and $\tilde{L}(x,u)$ given by (126) becomes

$$\begin{aligned}\tilde{L}(x,u) = {} & x^{\mathrm{T}}\left(R_1 + \sum_{k=2}^{r}(x^{\mathrm{T}}M_k x)^{k-1}\hat{R}_k\right)x \\ & + \left(\sum_{k=2}^{r}(x^{\mathrm{T}}M_k x)^{k-1}M_k x\right)^{\mathrm{T}}(S_{\mathrm{s}} - D_0R_0D^{\mathrm{T}})\left(\sum_{k=2}^{r}(x^{\mathrm{T}}M_k x)^{k-1}M_k x\right) + u^{\mathrm{T}}R_2u.\end{aligned}$$

6. Conclusion

A unified optimality-based framework was developed to address the problem of nonlinear–nonquadratic control for disturbance rejection of nonlinear systems with bounded exogenous disturbances. Specifically, by properly modifying a nonlinear–nonquadratic performance criterion to account for system disturbances the disturbance rejection problem was translated into an optimal control problem. Furthermore, the resulting optimal control law was shown to render the closed-loop nonlinear input–output map dissipative with respect to general supply rates and the Lyapunov function guaranteeing closed-loop stability was shown to be a solution to the steady-state Hamilton–Jacobi–Bellman equation and thus guaranteeing optimality. Finally, in the special case where

the controlled system is linear our results, with appropriate quadratic supply rates, specialize to the mixed-norm H_2/H_∞ and mixed H_2/positivity frameworks developed in [12] and [33, 34], respectively.

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