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Robust Stabilization for Discrete-time Systems with Slowly Time-varying Uncertainty

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ABSTRACT: *In this paper we construct a new class of parameter-dependent Lyapunov functions for discrete-time systems to guarantee robust stability in the presence of time-varying rate-restricted plant uncertainty. Extensions to a class of time-varying nonlinear uncertainty that generalizes the discrete-time multivariable Popov criterion are also considered. These results are then used for controller synthesis to address the problem of robust stabilization in the presence of slowly time-varying real parameters.* Copyright © 1996 Published by Elsevier Science Ltd

I. Introduction

Ever since Popov derived a frequency domain condition for the absolute stability of continuous-time nonlinear autonomous systems considerable effort has been devoted to deriving similar criteria for discrete-time systems. The literature on frequency domain absolute stability criteria is extensive [see (**1**) and references therein]. A convenient way of distinguishing these results is to focus on the allowable class of feedback nonlinearities. Specifically, the small gain, positivity, and circle theorems guarantee stability for arbitrarily time-varying nonlinearities whereas the Popov criterion does not. In order to develop refined absolute stability criteria for nonautonomous systems, the authors in (**2–6**) restrict consideration to sector-bounded rate-restricted time-varying nonlinearities. However, the authors in (**2–6**) deal exclusively with the continuous-time systems. In (**7, 8**) the authors consider a related absolute stability problem for single-input/single-output sampled-data systems wherein the nonlinear feedback element is a separable nonlinearity composed of a time-varying rate-restricted linear gain and a time-invariant nonlinearity. The main contribution of this paper is to develop refined absolute stability criteria for multiple-input/multiple-output discrete-time systems with rate-restricted time-varying nonlinearities. Since these stability criteria are generally stated in terms of a linear (nominal) system and apply to every element of a specified class of nonlinearities the present framework provides sufficient conditions for robust stability for a given class of time-varying rate-restricted uncertainty.

In the present paper we explicitly account for the maximum rate of time variation in the time-varying nonlinear element by considering a modified Lur'e-Postnikov Lyapunov function having the form

$$V(x) = x^T P x + \int_0^y \phi(\sigma, k)\,\mathrm{d}\sigma, \quad k = 1, 2, \ldots, \tag{1}$$

with the additional constraint

$$\int_0^y [\phi(\sigma, k+1) - \phi(\sigma, k)]\,\mathrm{d}\sigma \leqslant y^2 N_1 \tag{2}$$

where $y = Cx$ and $\phi(\cdot,\cdot)$ is a time-varying rate-restricted sector-bounded memoryless nonlinearity. A novel feature of Eq. (1) is that the expression for the Lyapunov difference $\Delta V(x)$ involves the (discrete) time rate of change of the nonlinearity. Thus, it turns out that satisfying the condition $\Delta V(x) < 0$ places a restriction on the maximum rate of time variation in the nonlinearity. Hence, within the context of linear uncertainty $\phi(y, k) = F(k)y$, the present framework provides robust stability conditions for discrete-time systems with slowly-varying real parameter uncertainty.

The contents of the paper are as follows. In Section II we establish definitions and several key lemmas. In Section III we present robust stability conditions for discrete-time systems with time-varying rate-restricted real parameter uncertainty. In Section IV we generalize the results of Section III to nonlinear uncertainty. Specifically, we consider an absolute stability problem for a class of time-varying rate-restricted sector-bounded memoryless nonlinearities. The main result given in Theorem 4.1 generalizes the discrete-time multivariable Popov criterion to slowly time-varying nonlinearities. In the single-input/single-output case Theorem 4.1 is a discrete-time analog of several well known absolute stability criteria for continuous-time systems with time-varying rate-restricted nonlinearities from the classical literature (**2–6**). Using the framework developed in Section III we proceed in Section V to give constructive sufficient conditions for robust stabilization for slowly time-varying real parameters via full-state feedback controllers. Finally, we close the paper in Section VI with conclusions.

II. Mathematical Preliminaries

In this section we establish definitions, notation, and several key lemmas. Let $\mathscr{R}$ and $\mathscr{C}$ denote the real and complex numbers, let $\mathbb{N}$ denote $\{1, 2, 3, \ldots\}$, let $()^T$ and $()^*$ denote transpose and complex conjugate transpose, let I_n or I denote the $n \times n$ identity matrix, and let 0_n denote the $n \times n$ zero matrix. Furthermore, $M \geqslant 0$ $(M > 0)$ denotes the fact that the Hermitian matrix M is nonnegative (positive) definite. An *asymptotically stable transfer function* is a transfer function each of whose poles is in the open unit disk. The space of asymptotically stable transfer functions is denoted by $\mathscr{R}H_\infty$, i.e. the real-rational subset of H_∞. Let

$$G(z) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

denote a state space realization of a transfer function $G(z)$, that is, $G(z) = C(zI-A)^{-1}B+D$. The notation '$\overset{\min}{\sim}$' is used to denote a minimal realization. In addition, the parahermitian conjugate $G^{\sim}(z)$ of $G(z)$ has the realization

$$G^{\sim}(z) \sim \left[\begin{array}{c|c} A^{-T} & A^{-T}C^{T} \\ \hline -B^{T}A^{-T} & D^{T}-B^{T}A^{-T}C^{T} \end{array}\right].$$

A square transfer function $G(z)$ is called *positive real* (**9**) if: (1) all poles of $G(z)$ are in the closed unit disk; and (2) $G(z)+G^{*}(z)$ is nonnegative definite for $|z| > 1$. A square transfer function $G(z)$ is called *strictly positive real* (**10, 11**) if: (1) $G(z)$ is asymptotically stable; and (2) $G(e^{j\theta})+G^{*}(e^{j\theta})$ is positive definite for all $\theta \in [0, 2\pi]$. Note that a minimal realization of a positive real transfer function is stable in the sense of Lyapunov, while a minimal realization of a strictly positive real transfer function is asymptotically stable.

For notational convenience we will omit all matrix dimensions throughout the paper and assume that all quantities have compatible dimensions. Furthermore, in this paper, $G(z)$ will denote an $m \times m$ transfer function with input $u \in \mathscr{R}^{m}$, output $y \in \mathscr{R}^{m}$, and internal state $x \in \mathscr{R}^{n}$. Next, we state the discrete-time positive real lemma used to characterize positive realness in the state-space setting.

Lemma 1 [positive real lemma (10)].

$$G(z) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

is positive real if and only if there exist matrices P, L, and W with P positive definite such that

$$P = A^{T}PA + L^{T}L, \tag{3}$$

$$0 = B^{T}PA - C + W^{T}L, \tag{4}$$

$$0 = D + D^{T} - B^{T}PB - W^{T}W. \tag{5}$$

Next, we show that if $D+D^{T}-B^{T}PB > 0$, where P satisfies Eqs (3)–(5) then Eqs (3)–(5) specialize to a single Riccati equation for characterizing positive realness. For the statement of this result recall that a square transfer function $G(z)$ is *strongly positive real* if it is strictly positive real and $D+D^{T} > 0$ (**1, 12**).

Lemma 2 [strong positive real lemma (1)].

Let

$$G(z) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then $G(z)$ is strongly positive real if and only if there exists positive definite matrices P and R such that

$$D + D^{T} - B^{T}PB > 0, \tag{6}$$

$$P = A^T PA + (B^T PA - C)^T (D + D^T - B^T PB)^{-1} (B^T PA - C) + R. \tag{7}$$

III. Robust Stability for Discrete-time Systems with Slowly-varying Real Parameter Uncertainty

In this section we address the robust stability problem for discrete-time systems with time-varying rate-restricted real parameter uncertainty. Specifically, consider

$$x(k+1) = (A + \Delta A(k))x(k), \quad k \in \mathbb{N} \tag{8}$$

where $\Delta A(\cdot) \in \mathcal{U}$ and $\mathcal{U}$ is the uncertainty set defined by

$$\mathcal{U} \triangleq \{\Delta A(\cdot) : \Delta A(k) = -B_0 F(k) C_0, \quad F(\cdot) \in \mathcal{F}\} \tag{9}$$

where $\mathcal{F}$ satisfies

$$\mathcal{F} \triangleq \{F : \mathbb{N} \to \mathcal{R}^{m \times m} : 2F^T(k) M^{-1} F(k) \leqslant F(k) + F^T(k)$$
$$F(k+1) - F(k) \leqslant N_1, \quad k \in \mathbb{N}\}, \tag{10}$$

and where $B_0 \in \mathcal{R}^{n \times m}$, $C_0 \in \mathcal{R}^{m \times n}$ are fixed matrices denoting the structure of the uncertainty, $F(k) \in \mathcal{R}^{m \times m}$ is an uncertain matrix, $M \in \mathcal{R}^{m \times m}$ is a given positive definite matrix, and N_1 is a given $m \times m$ matrix.

Remark 3.1

Note that the condition

$$F(k+1) - F(k) \leqslant N_1 \tag{11}$$

in Eq. (10) provides a measure of the maximum rate of time variation in the uncertain real parameters.

Remark 3.2

In the case where $F(k)$, $k \in \mathbb{N}$, is symmetric it can be shown that $2F^T(k) M^{-1} F(k) \leqslant F(k) + F^T(k)$ if and only if $0 \leqslant F(k) \leqslant M$ (**12**).

Next, we define a set $\mathcal{N}$ such that the product of the transpose of every matrix in $\mathcal{N}$ and every matrix in $\mathcal{F}$ is nonnegative definite by

$$\mathcal{N} \triangleq \{N \in \mathcal{R}^{m \times m} : N^T F(k) = F^T(k) N \geqslant 0, \quad F(\cdot) \in \mathcal{F}\}. \tag{12}$$

Remark 3.3

Since M in Eq. (10) is positive definite there exists an $m \times m$ nonnegative definite matrix μ such that (**12**)

$$F^T(k) N \leqslant \mu, \quad F(\cdot) \in \mathcal{F}. \tag{13}$$

For convenience we shall say that $A + \Delta A(\cdot)$ is asymptotically stable if the zero solution of the time-varying system $x(k+1) = (A + \Delta A(k))x(k)$ is asymptotically stable. The next result provides a sufficient condition for the robust stability of Eq. (8) for all $\Delta A(\cdot) \in \mathcal{U}$. For convenience in stating this result define the notation

$$\hat{R} \triangleq (M^{-1}+NC_0B_0)+(M^{-1}+NC_0B_0)^T-B_0^TPB_0$$
$$-B_0^TC_0^T\mu C_0B_0-B_0^TC_0^TN_1^TNC_0B_0, \quad (14)$$

$$\hat{B} \triangleq B_0^TPA-C_0-NC_0(A-I)+B_0^TC_0^T\mu C_0(A-I)+B_0^TC_0^TN_1^TNC_0A \quad (15)$$

for arbitrary $P\in\mathscr{R}^{n\times n}$.

Theorem I

Let N_1 be a given $m\times m$ matrix, assume $N\in\mathcal{N}$, and assume N_1^TN and μ are nonnegative definite. Furthermore, assume that there exist $n\times n$ positive-definite matrices P and R such that

$$\hat{R}>0, \quad (16)$$

$$P=A^TPA+(A-I)^TC_0^T\mu C_0(A-I)+A^TC_0^TN_1^TNC_0A+\hat{B}^T\hat{R}^{-1}\hat{B}+R. \quad (17)$$

Then, for all $\Delta A(\cdot)\in\mathscr{U}$, the uncertain system [Eq. (8)] is asymptotically stable with the Lyapunov function

$$V(x)=x^T[P+C_0^TN^TF(k)C_0]x. \quad (18)$$

■

Proof: In order to prove the asymptotic stability of Eq. (8) for all $\Delta A(\cdot)\in\mathscr{U}$, consider the Lyapunov function candidate given by Eq. (18). Since P is positive definite and $N\in\mathcal{N}$, it follows that $V(x)$ defined by Eq. (18) is positive definite for all nonzero x. The corresponding Lyapunov difference is given by

$$\Delta V(x(k))=x^T(k+1)[P+C_0^TN^TF(k+1)C_0]x(k+1)$$
$$-x^T(k)[P+C_0^TN^TF(k)C_0]x(k) \quad (19)$$

or, equivalently, using Eq. (8)

$$\Delta V(x(k))=x^T(k)[A^TPA-P-C_0^TF^T(k)B_0^TPA$$
$$-A^TPB_0F(k)C_0+C_0^TF^T(k)B_0^TPB_0F(k)C_0]x(k)$$
$$+x^T(k+1)C_0^TN^TF(k+1)C_0x(k+1)-x^T(k)C_0^TN^TF(k)C_0x(k). \quad (20)$$

Next, adding and subtracting $x^T(k+1)C_0^TN^TF(k)C_0x(k+1)$ to and from Eq. (20) yields

$$\Delta V(x(k))=x^T(k)[A^TPA-P-C_0^TF^T(k)B_0^TPA$$
$$-A^TPB_0F(k)C_0+C_0^TF^T(k)B_0^TPB_0F(k)C_0]x(k)$$
$$+x^T(k+1)C_0^TN^T[F(k+1)-F(k)]C_0x(k+1)$$
$$+x^T(k+1)C_0^TN^TF(k)C_0x(k+1)-x^T(k)C_0^TN^TF(k)C_0x(k). \quad (21)$$

Now using the fact that $F(k+1)-F(k)\leqslant N_1$, $k\in\mathbb{N}$, and adding and subtracting $2x^T(k)C_0^TN^TF(k)C_0x(k+1)$ and $x^T(k)C_0^TN^TF(k)C_0x(k)$ to and from Eq. (21) yields

$$\Delta V(x(k)) \leqslant x^T(k)[A^TPA-P-C_0^TF^T(k)B_0^TPA$$
$$-A^TPB_0F(k)C_0+C_0^TF^T(k)B_0^TPB_0F(k)C_0]x(k)$$
$$+x^T(k+1)C_0^TN_1^TNC_0x(k+1)+2x^T(k)C_0^TF^T(k)NC_0[x(k+1)-x(k)]$$
$$+[x(k+1)-x(k)]^TC_0^TN^TF(k)C_0[x(k+1)-x(k)]. \qquad (22)$$

Next, adding and subtracting $x^T(k)C_0^T[2F^T(k)M^{-1}F(k)-F(k)-F^T(k)]C_0x(k)$ to and from Eq. (22), using Eq. (13), and grouping terms in Eq. (22) becomes

$$\Delta V(x(k)) \leqslant x^T(k)[[A^TPA-P+(A-I)^TC_0^T\mu C_0(A-I)+A^TC_0^TN_1^TNC_0A]$$
$$-C_0^TF^T(k)[B_0^TPA-C_0-NC_0(A-I)+B_0^TC_0^T\mu C_0(A-I)+B_0^TC_0^TN_1^TNC_0A]$$
$$-[B_0^TPA-C_0-NC_0(A-I)+B_0^TC_0^T\mu C_0(A-I)+B_0^TC_0^TN_1^TNC_0A]^TF(k)C_0$$
$$-C_0^TF^T(k)[(M^{-1}+NC_0B_0)+(M^{-1}+NC_0B_0^T-B_0^TPB_0$$
$$-B_0^TC_0^T\mu C_0B_0-B_0^TC_0^TN_1^TNC_0B_0]F(k)C_0]x(k)$$
$$+x^T(k)C_0^T[2F^T(k)M^{-1}F(k)-F(k)-F^T(k)]C_0x(k) \qquad (23)$$

or, equivalently, using Eqs (14)–(17)

$$\Delta V(x(k)) \leqslant -x^T(k)[R+w^Tw]x(k)+x^T(k)C_0^T[2F^T(k)M^{-1}F(k)$$
$$-F(k)-F^T(k)]C_0x(k) \qquad (24)$$

where

$$w \triangleq \hat{R}^{-1/2}\hat{B}+\hat{R}^{1/2}F(k)C_0 \qquad (25)$$

Since R is positive definite and $F(\cdot)\in\mathcal{F}$ it follows that $\Delta V(x)$ is negative definite. Hence Eq. (8) is asymptotically stable for all $\Delta A(\cdot)\in\mathcal{U}$. ■

Remark 3.4

Note that the Lyapunov function $V(x)$ given by Eq. (18) is explicitly dependent on the uncertain parameters $F(k)$. In the terminology of (**12, 13**), this is a parameter-dependent Lyapunov function. Since the parameter-dependent Lyapunov function $V(x)$ explicitly depends on the uncertain *time-varying* parameters $F(k)$ it enforces an *a priori* restriction on the allowable time-variation of the uncertain parameters. Specifically, if $F(k)$ were permitted to be arbitrarily time-varying then the terms involving $F(k+1)-F(k)$ would potentially subvert the negative definiteness of $\Delta V(x)$.

Remark 3.5

In the case where the only information available about the rate of time variation of the uncertainty is $F(k+1)-F(k) \leqslant 0$ asymptotic stability of Eq. (8) can be guaranteed with $N_1 = 0$ in Theorem I.

Remark 3.6

The existence of the Riccati Eq. (17) can be guaranteed by invoking a strong positive real condition on

$$\mathcal{G}(z) \triangleq M^{-1}+(I+(z-1)N)G(z)-\tfrac{1}{2}[|z-1|^2G^{\sim}(z)\mu G(z)+G^{\sim}(z)N_1^T NG(z)] \qquad (26)$$

where

$$G(z) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B_0 \\ \hline C_0 & 0 \end{array}\right].$$

Specifically, $\mathcal{G}(z)$ is strictly positive real if and only if there exist matrices P, L, and W with P positive definite such that

$$P = A^TPA+(A-I)^TC_0^T\mu C_0(A-I)+A^TC_0^TN_1^TNC_0A+L^TL \qquad (27)$$

$$0 = B_0^TPA-C_0-NC_0(A-I)+B_0^TC_0^T\mu C_0(A-I)+B_0^TC_0^TN_1^TNC_0A+W^TL, \qquad (28)$$

$$0 = (M^{-1}+NC_0B_0)+(M^{-1}+NC_0B_0)^T-B_0^TPB_0-B_0^TC_0^T\mu C_0B_0 - B_0^TC_0^TN_1^TNC_0B_0-W^TW \qquad (29)$$

are satisfied, the pair (A, L) is observable, and rank $\hat{G}(z) = m$, for $z = e^{j\omega}, \omega \in \mathcal{R}$, where

$$\hat{G}(z) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B_0 \\ \hline L & W \end{array}\right]. \qquad (30)$$

However, Eqs (27)–(29) along with $\hat{R} > 0$ are equivalent to the Riccati Eq. (17). Hence, strong positive realness of $\mathcal{G}(z)$ guarantees the existence of positive definite matrices P and R satisfying Eq. (17).

IV. Absolute Stability Criteria for Discrete-time Systems with Slowly-varying Nonlinearities

In this section we provide a generalization of the results of Section III to nonlinear uncertainty. Specifically, we consider the absolute stability problem for a class Φ of time-varying rate-restricted sector-bounded nonlinearities $\phi : \mathcal{R}^m \times \mathbb{N} \to \mathcal{R}^m$. Now, given

$$G(z) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

we derive conditions that guarantee global asymptotic stability of the negative feedback interconnection of $G(z)$ and $\phi(\cdot,\cdot)$ for all $\phi(\cdot,\cdot) \in \Phi$. Note that the negative feedback interconnection of $G(z)$ and $\phi(\cdot,\cdot)$ has the state-space representation

$$x(k+1) = Ax(k) - B\phi(y,k), \quad k \in \mathbb{N} \tag{31}$$

$$y(k) = Cx(k). \tag{32}$$

To state the main result of this section, the following definitions are needed. Let $M \in \mathscr{R}^{m\times m}$ be a given positive definite diagonal matrix. Next, define the set Φ of allowable time-varying rate-restricted sector-bounded nonlinearities $\phi(\cdot,\cdot)$ by

$$\Phi \triangleq \{\phi : \mathscr{R}^m \times \mathbb{N} \to \mathscr{R}^m : \phi(y,k) = [\phi_1(y_1,\cdot),\ldots,\phi_m(y_m,\cdot)],$$

$$\phi^T(y,k)[M^{-1}\phi(y,k) - y] \leqslant 0,$$

$$0 < \frac{\phi_i(\sigma_1,k)-\phi_i(\sigma_2,k)}{\sigma_1-\sigma_2} < \hat{\mu}_i, \quad i = 1,\ldots,m, \quad \sigma_1,\sigma_2 \in \mathscr{R}, \quad \sigma_1 \neq \sigma_2,$$

$$2\sum_{i=1}^{m}\int_0^{y_i}[\phi_i(\sigma,k+1)-\phi_i(\sigma,k)]\,\mathrm{d}\sigma \leqslant y^T N_1 y, \quad y \in \mathscr{R}^m, \quad k \in \mathbb{N}\} \tag{33}$$

where $y_i = C_i x$ and C_i denotes the ith row of C.

Remark 4.1

Note that the conditions characterizing Φ are implied by

$$0 \leqslant \phi_i(y_i,k)y_i \leqslant M_{ii}y_i^2, \quad y_i \in \mathscr{R}, \quad i = 1,\ldots,m, \quad k \in \mathbb{N} \tag{34}$$

and

$$2\int_0^{y_i}[\phi_i(\sigma,k+1)-\phi_i(\sigma,k)]\,\mathrm{d}\sigma \leqslant y_i^2 N_{1_{i,i}} y_i \in \mathscr{R}, \quad i = 1,\ldots,m, \quad k \in \mathbb{N}. \tag{35}$$

Furthermore, Eq. (35) provides a measure of the maximum rate of time variation of the nonlinearity $\phi(y,k)$.

For convenience in stating the next result define the notation

$$\bar{R} \triangleq (M^{-1}+NCB)+(M^{-1}+NCB)^T - B^TPB - B^TC^T\hat{\mu}NCB - B^TC^TN_1NCB, \tag{36}$$

$$\bar{B} \triangleq B^TPA - C - NC(A-I) + B^TC^T\hat{\mu}NC(A-I) + B^TC^TN_1NCA \tag{37}$$

for arbitrary $P \in \mathscr{R}^{n\times n}$.

Theorem II

Let

$$G(z) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

and let N_1, N, and $\hat{\mu}$ be given $m \times m$ diagonal matrices. Furthermore, assume N_1, N, are nonnegative definite and $\hat{\mu} > 0$. Then

$$\mathscr{G}(z) \triangleq M^{-1} + [I + (z-1)N]G(z) - \tfrac{1}{2}[|z-1|^2 G^{\sim}(z)\hat{\mu}NG(z) + G^{\sim}(z)N_1 NG(z)] \tag{38}$$

is strongly positive real if and only if there exist $n \times n$ positive-definite matrices P and R such that

$$\bar{R} > 0, \tag{39}$$

$$P = A^T PA + (A-I)^T C^T \hat{\mu} NC(A-I) + A^T C^T N_1 NCA + \bar{B}^T \bar{R}^{-1} \bar{B} + R. \tag{40}$$

■

In this case,

$$V(x) = x^T Px + 2 \sum_{i=1}^{m} \int_0^{y_i} \phi_i(\sigma, k) N_{ii} \, \mathrm{d}\sigma \tag{41}$$

is a Lyapunov function that guarantees that the negative feedback interconnection of $G(z)$ and $\phi(\cdot,\cdot)$ is asymptotically stable for all $\phi(\cdot,\cdot) \in \Phi$.

Proof: If $\mathscr{G}(z)$ is strongly positive real, it follows from spectral factorization theory that there exists a spectral factor $\mathscr{M}(z)$ such that $\mathscr{G}(z) + \mathscr{G}^{\sim}(z) = \mathscr{M}^{\sim}(z)\mathscr{M}(z)$, $z = e^{j\omega}$, where $\mathscr{M}^{\pm}(z) \in \mathscr{R}H_{\infty}$. The existence of positive definite matrices P and R satisfying Eqs (39) and (40) now follows from algebraic state-space realization manipulations.

Conversely, to prove that Eqs (39) and (40) imply that $\mathscr{G}(z)$ is strongly positive real, add and subtract $A^T PA$, $A^T Pz$, and $\bar{z}PA$ to and from Eq. (40) so that

$$\begin{aligned} 0 = {} & (\bar{z}I - A)^T PA + A^T P(zI - A) + (\bar{z}I - A)^T P(zI - A) \\ & - (A-I)^T C^T \hat{\mu} NC(A-I) - A^T C^T N_1 NCA - \bar{B}^T \bar{R}^{-1} \bar{B} - R, \quad z = e^{j\omega}. \end{aligned} \tag{42}$$

Next, forming $B^T(\bar{z}I - A)^{-T}$ (Eq. (42)) $(zI - A)^{-1}B$ and using Eq. (37) we obtain

$$\begin{aligned} 0 = {} & [\bar{B} + C + NC(A-I) - B^T C^T \hat{\mu} NC(A-I) - B^T C^T N_1 NCA](zI - A)^{-1} B \\ & + B^T(\bar{z}I - A)^{-T}[\bar{B} + C + NC(A-I) - B^T C^T \hat{\mu} NC(A-I) - B^T C^T N_1 NCA]^T \\ & + B^T PB - B^T(\bar{z}I - A)^{-T}[(A-I)^T C^T \hat{\mu} NC(A-I) + A^T C^T N_1 NCA \\ & + \bar{B}^T \bar{R}^{-1} \bar{B}](zI - A)^{-1} B - W^*(z) W(z), \quad z = e^{j\omega} \end{aligned} \tag{43}$$

where $W(z) \triangleq R^{1/2}(zI - A)^{-1}B$. Adding and subtracting $\bar{R}$ to and from Eq (43), using Eq. (36), and grouping terms yields

$$\mathscr{G}(z) + \mathscr{G}^*(z) = W^*(z) W(z) + U^*(z) U(z) > 0, \quad z = e^{j\omega} \tag{44}$$

where $U(z) \triangleq \bar{R}^{-1/2} - \bar{R}^{-1/2}\bar{B}(zI - A)^{-1}B$. Hence $\mathscr{G}(z)$ is strongly positive real.

Next, for $\phi(\cdot,\cdot) \in \Phi$ consider the Lyapunov function candidate (41). First note that since P is positive definite and $\phi(\cdot,\cdot) \in \Phi$, $V(x)$ defined by Eq. (41) is positive for all nonzero x. Thus, the corresponding Lyapunov difference is given by

$$\Delta V(x(k)) = x^T(k)[A^TPA-P]x(k)-\phi^T(y,k)B^TPAx(k)$$
$$-x^T(k)A^TPB\phi(y,k)+\phi^T(y,k)B^TPB\phi(y,k)$$
$$+2\sum_{i=1}^{m}\left[\int_0^{y_i(k+1)}\phi_i(\sigma,k+1)N_{ii}\,\mathrm{d}\sigma-\int_0^{y_i(k)}\phi_i(\sigma,k)N_{ii}\,\mathrm{d}\sigma\right]. \quad (45)$$

Adding and subtracting

$$2\sum_{i=1}^{m}\int_0^{y_i(k-1)}\phi_i(\sigma,k)N_i\,\mathrm{d}\sigma \quad \text{and} \quad 2\phi^T(y,k)[M^{-1}\phi(y,k)-y]$$

to and from Eq. (45) and grouping terms yields

$$\Delta V(x(k)) = x^T(k)[A^TPA-P]x(k)-\phi^T(y,k)[B^TPA-C]x(k)$$
$$-x^T(k)[B^TPA-C]^T\phi(y,k)-\phi^T(y,k)[2M^{-1}-B^TPB]\phi(y,k)$$
$$+2\sum_{i=1}^{m}\int_0^{y_i(k+1)}[\phi_i(\sigma,k+1)-\phi_i(\sigma,k)]N_{ii}\,\mathrm{d}\sigma$$
$$+2\sum_{i=1}^{m}\int_{y_i(k)}^{y_i(k+1}\phi_i(\sigma,k)N_{ii}\,\mathrm{d}\sigma+2\phi^T(y,k)[M^{-1}\phi(y,k)-y]. \quad (46)$$

Next, using the fact that

$$0<\frac{\phi_i(\sigma_1,k)-\phi_i(\sigma_2,k)}{\sigma_1-\sigma_2}<\hat{\mu}_i, \quad i=1,\ldots,m,$$

it follows from the mean value theorem that

$$2\sum_{i=1}^{m}\int_{y_i(k)}^{y_i(k+1)}\phi_i(\sigma,k)N_{ii}\,\mathrm{d}\sigma \leqslant 2\phi^T(y,k)N[y(k+1)-y(k)]$$
$$+[y(k+1)-y(k)]^T\hat{\mu}N[y(k+1)-y(k)].$$

Now, since $\phi(\cdot,\cdot)\in\Phi$ and $y(k+1)-y(k) = C(A-I)x(k)-CB\phi(y,k)$, Eq. (46) becomes

$$\Delta V(x(k)) \leqslant x^T(k)[A^TPA-P+(A-I)^TC^T\hat{\mu}NC(A-I)+A^TC^TN_1NCA]x(k)$$
$$-\phi^T(y,k)[B^TPA-C-NC(A-I)+B^TC^T\hat{\mu}NC(A-I)+B^TC^TN_1NCA]x(k)$$
$$-x^T(k)[B^TPA-C-NC(A-I)+B^TC^T\hat{\mu}NC(A-I)+B^TC^TN_1NCA]^T\phi(y,k)$$
$$-\phi^T(y,k)[(M^{-1}+NCB)+(M^{-1}+NCB)^T-B^TPB-B^TC^T\hat{\mu}NCB$$
$$-B^TC^TN_1NCB]\phi(y,k)+2\phi^T(y,k)[M^{-1}\phi(y,k)-y] \quad (47)$$

or, equivalently, using [Eqs (36), (37) and (40)]

$$\Delta V(x) \leqslant -x^T Rx - v^T v + 2\phi^T(y,k)[M^{-1}\phi(,k) - y] \tag{48}$$

where

$$v \triangleq \bar{R}^{-1/2}\bar{B}x(k) + \bar{R}^{1/2}\phi(y,k).$$

Since R is positive definite and $\phi^T(y,k)[M^{-1}\phi(y,k)-y] \leqslant 0$ it follows that $\Delta V(x)$ is negative definite. Hence global asymptotic stability of feedback interconnection of $G(z)$ and $\phi(\cdot,\cdot)$ is established for all $\phi(\cdot,\cdot) \in \Phi$. ■

Remark 4.2

A similar proof using the three equation form (27)–(29) with B_0, C_0 replaced by B, C, respectively, in the positive real condition Eq. (38) can also be constructed. In this case the condition (38) in Theorem II can be relaxed from strongly positive real to strictly positive real.

Remark 4.3

Theorem II is a discrete-time analog of a similar result by Rekasius and Rowland (**4**) and Srinath *et al.* (**5**) for single-input/single-output continuous-time systems. Furthermore, Theorem II generalizes the results of (**8**) which only considers single-input/single-output systems with a separable time-varying nonlinearity to a more general class of rate-restricted nonlinearities.

Remark 4.4

In the special case where $\phi(y,k) = F(k)y$, Theorem II specializes to Theorem I for the case of diagonal time-varying rate-restricted uncertainty $F(k)$. However, the results of Section III allow for fully coupled uncertain rate-restricted uncertainties $F(k)$ which cannot be addressed by means of the nonlinear theory.

Remark 4.5

Setting $N_1 = 0$ in Eq. (38) Theorem II specializes to the discrete-time multivariable Popov criterion (**1, 12**).

Remark 4.6

It is interesting to note that in the case where the only information available about the rate of the time variation of the nonlinearity is

$$\sum_{i=1}^{m} \int_0^{y_i} [\phi_i(\sigma,k+1) - \phi_i(\sigma,k)]\, d\sigma \leqslant 0 \tag{49}$$

asymptotic stability of Eqs (31) and (32) can be guaranteed with $N_1 = 0$ in Theorem II. In this case $\mathcal{G}(z)$ given by Eq. (38) specializes to

$$\mathcal{G}^P(z) \triangleq M^{-1} + (I + (z-1)N)G(z) - \tfrac{1}{2}|z-1|^2 G^{\sim}(z)\hat{\mu}NG(z). \tag{50}$$

Furthermore, in the special case where

$$\phi_i(y_i,k) = f_i(k)\hat{\phi}_i(y_i), \quad i = 1,\ldots,m \tag{51}$$

where $f_i(k)$ is a linear time-varying gain and $\hat{\phi}_i(y_i)$ is a first and third quadrant time-invariant sector-bounded memoryless nonlinearity, Eq. (49) is satisfied if $f_i(k+1)-f_i(k) \leqslant 0$. Hence, for this class of time-varying rate-restricted nonlinearities the discrete-time Popov criterion (**14**) provides sufficient conditions for absolute stability.

V. Robust Stabilization for Discrete-time Systems with Slowly-varying Real Parameter Uncertainty

In this section we consider the robust stabilization problem for systems with time-varying rate-restricted real parameter uncertainty. The problem involves the set $\mathscr{U}$ given by Eq. (9) of uncertain perturbations $\Delta A(\cdot)$ of the nominal (A, B) system. The goal of the robust stabilization problem is to determine a state feedback controller that stabilizes the plant for all variations in $\mathscr{U}$.

Robust Stabilization Problem

Determine $K \in \mathscr{R}^{m\times n}$ such that the closed-loop system consisting of the nth-order controlled plant

$$x(k+1) = (A+\Delta A(k))x(k)+Bu(k), \quad k \in \mathbb{N} \tag{52}$$

and the state feedback controller

$$u(k) = Kx(k) \tag{53}$$

is asymptotically stable for all $\Delta A(\cdot) \in \mathscr{U}$.

For each uncertain variation $\Delta A(\cdot) \in \mathscr{U}$, the closed-loop system can be written as

$$x(k+1) = (A+BK+\Delta A(k))x(k), \quad k \in \mathbb{N}. \tag{54}$$

The following result gives a sufficient condition for constructing a state feedback gain K that solves the robust stabilization problem with time-varying rate-restricted real parameter uncertainty. For the statement of this result let R_1 and R_2 be $n\times n$ and $m\times m$ positive definite matrices, respectively. Furthermore, for notational convenience recall the definitions of $\hat{R}$ and $\hat{B}$ and define

$$S \triangleq B_0^T PB-NC_0B+B_0^T C_0^T \mu C_0B+B_0^T C_0^T N_1^T NC_0B,$$

$$P_a \triangleq B^TPA+B^TC_0^T \mu C_0(A-I)+B^TC_0^T N_1^T NC_0A+S^T\hat{R}^{-1}\hat{B},$$

$$R_{2a} \triangleq R_2+B^TPB+B^TC_0^T \mu C_0B+B_0^T C_0^T N_1^T NC_0B+S^T\hat{R}^{-1}S$$

for arbitrary $P \in \mathscr{R}^{n\times n}$.

Theorem III

Assume $\mu \geqslant 0$, $N \in \mathcal{N}$, and $N_1^T N \geqslant 0$. Furthermore, suppose there exists an $n\times n$ positive-definite matrix P satisfying

$$\hat{R} > 0, \tag{55}$$

$$P = A^T PA + (A-I)^T C_0^T \mu C_0 (A-I) + A^T C_0^T N_1^T N C_0 A + R_1 + \hat{B}^T \hat{R}^{-1} \hat{B} - P_a^T R_{2a}^{-1} P_a \quad (56)$$

and let K be given by

$$K = -R_{2a}^{-1} P_a. \quad (57)$$

Then $A + BK + \Delta A(\cdot)$ is asymptotically stable for all $\Delta A(\cdot) \in \mathcal{U}$. ■

Proof: With K given by Eq. (57), it follows that Eq. (56) is equivalent to

$$\begin{aligned} P = {} & (A+BK)^T P(A+BK) + R_1 + ((A+BK)-I)^T C_0^T \mu C_0((A+BK)-I) \\ & + (A+BK)^T C_0^T N_1^T N C_0 (A+BK) + [B_0^T P(A+BK) - C_0 N C_0((A+BK)-I) \\ & + B_0^T C_0^T \mu C_0((A+BK)-I) + B_0^T C_0^T N_1^T N C_0(A+BK)]^T \hat{R}^{-1} \\ & \cdot [B_0^T P(A+BK) - C_0 - N C_0((A+BK)-I) + B_0^T C_0^T \mu C_0((A+BK)-I) \\ & + B_0^T C_0^T N_1^T N C_0(A+BK)]. \end{aligned} \quad (58)$$

It now follows from Theorem I that $A + BK + \Delta A(\cdot)$ is asymptotically stable for all $\Delta A(\cdot) \in \mathcal{U}$. ■

Remark 5.1

Theorem III presents sufficient conditions for designing robust full-state feedback controllers for time-varying rate-restricted real parameter uncertainty. Using the fixed-structure controller synthesis framework developed in (**12**, **13**) these results can be readily extended to fixed-order (i.e. full and reduced-order) dynamic compensation.

VI. Conclusion

In this paper we developed a new class of parameter-dependent Lyapunov functions for discrete-time systems that explicitly depend on the time variation of the uncertain parameters to guarantee robust stability for discrete-time systems in the presence of time-varying rate-restricted plant uncertainty. Extensions to a class of time-varying nonlinear uncertainty that generalizes the discrete-time multivariable Popov criterion were also developed. Finally, using the parameterized Lyapunov function framework developed in the analysis part of the paper constructive sufficient conditions for robust controller synthesis for slowly time-varying real parameters via full-state feedback were obtained.

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