

Since for square matrices the trace is merely the sum of the diagonal elements, $\text{tr}YUU^*Y^*D^*D = \sum_{i=1}^d (YUU^*Y^*)_{ii}d_i$, where $\{d_i\}_1^d$ are the positive diagonal elements of D^*D . Thus

$$\begin{aligned} \text{tr}YUU^*Y^*D^*D &\leq \sum_{i=1}^d |(YUU^*Y^*)_{ii}|d_i \\ &\leq \max_{i=1,2,\dots,d} |(YUU^*Y^*)_{ii}| \text{tr}D^*D. \end{aligned} \quad (10)$$

Since the magnitude of each element of YUU^*Y^* is bounded above by $|YUU^*Y^*|$, (i.e., the maximum singular value of YUU^*Y^*), and Y unitary then, from (10), we deduce

$$\begin{aligned} \|MU\|_F^2 &= \text{tr}YUU^*Y^*D^*D \leq |U|^2 |\text{tr}D^*D| \\ &= |U|^2 \sum_{i=1}^n \sigma_i^2 \end{aligned} \quad (11)$$

$$\Rightarrow \|MU\|_F \leq \sqrt{\sum_{i=1}^n \sigma_i^2} |U| \quad (12)$$

$$\Rightarrow \|M\| \leq \sqrt{\sum_{i=1}^n \sigma_i^2}. \quad (13)$$

Equality in (11) can be achieved by putting $U = I$.

A consequence of Proposition 1 is that the H^2 -norm is also induced. To see this let $L_{d \times q}^\infty$ be the space of $d \times q$ matrix-valued essentially bounded functions defined on the $j\omega$ axis, endowed with the norm

$$\|f\|_\infty := \text{ess sup}_{\omega \in \mathbb{R}} |f(j\omega)|, \quad f \in L_{d \times q}^\infty. \quad (14)$$

Likewise let $L_{q \times q}^2$ be the space of $q \times q$ matrix-valued Lebesgue square integrable functions defined on the $j\omega$ -axis, under the norm

$$\|g\|_2 := \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \|g\|_F^2 d\omega}, \quad g \in L_{q \times q}^2. \quad (15)$$

Letting $G \in H_{q \times d}^2$, the multiplication operator associated with G and acting from $L_{d \times q}^\infty$ into $L_{q \times q}^2$ is denoted by M_G . More precisely, we show in the following proposition that if $M_G f = Gf$, for $f \in L_{d \times q}^\infty$, then the operator induced norm of M_G is equal to $\|G\|_2$.

Proposition 2: The operator induced norm of M_G , as a multiplication operator from $L_{d \times q}^\infty$ into $L_{q \times q}^2$, is equal to $\|G\|_2$, that is

$$\|M_G\| := \sup_{\substack{\|f\|_\infty \leq 1 \\ f \in L_{d \times q}^\infty}} \|Gf\|_2 = \|G\|_2. \quad (16)$$

Proof: Let $g \in L_{d \times q}^\infty$, $\|g\|_\infty \leq 1$ we have that

$$\begin{aligned} \|M_G g\|_2^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}\{(Mg)(Mg)^*\}(j\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}\{M^* M g g^*\}(j\omega) d\omega \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(j\omega)|^2 \text{tr}\{M^* M\}(\omega) d\omega \\ &\leq \|g\|_\infty^2 \|M\|_2^2. \end{aligned} \quad (17)$$

Equality follows by choosing $g = I$, the identity matrix.

Remark: Propositions 1 and 2 show that the Frobenius and the H^2 norms are induced, when allowed to act on special spaces of matrices and matrix-valued functions, respectively. In [1], only standard spaces of vectors \mathbb{C}^n have been considered.

It should be noted that the Frobenius norm is also induced on the space of linear functionals on the space of matrices, say $\mathbb{C}^{m \times n}$, as a Hilbert space with the usual matrix inner product, i.e., for $A, B \in \mathbb{C}^{m \times n}$, $\langle A, B \rangle = \text{tr}AB^*$, since it is isometrically isomorphic to its dual space. Same observation applies to the H^2 norm, which is induced on the space of bounded linear functionals on the space $L_{m \times n}^2$ with the inner product, for $f, g \in L_{m \times n}^2$, $\langle f, g \rangle = 1/2\pi \int_{-\infty}^{\infty} \text{tr}fg^*(j\omega) d\omega$, for the same reason.

REFERENCES

- [1] V. Y. Chellaboina and W. M. Haddad, "Is the Frobenius matrix norm induced?," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 2137–2139, Dec. 1995.

Authors' Reply

VijaySekhar Chellaboina and Wassim M. Haddad

In [1], we showed that the Frobenius norm is not induced from $(\mathbb{C}^n, \|\cdot\|')$ into $(\mathbb{C}^m, \|\cdot\|'')$, where $m, n > 1$, and $\|\cdot\|'$ and $\|\cdot\|''$ are *vector norms*. In [2], the author shows that if $\|\cdot\|'$ and $\|\cdot\|''$ are taken to be *matrix norms*, then the Frobenius norm is indeed induced. Furthermore, the author shows that the H_2 norm is also induced from an H_∞ space into an H_2 space. In this note, we show that even though the results of [2] are interesting and provide additional insight into the problem addressed in [1], they are not surprising and are in fact a very special case of a general result on bounded linear operators. In particular, in this communique we show that *every* operator norm defined on a normed linear space is induced on particular normed linear spaces. As a special case of this result, we show that *every* matrix norm is induced on particular *matrix* spaces.

The notation we use in this communique is fairly standard. Specifically, \mathbb{R} denotes the set of real numbers, \mathbb{C} denotes the set of complex numbers, \mathbb{R}^n (respectively, \mathbb{C}^n) denotes the set of n -dimensional real (respectively, complex) vectors, and $\mathbb{R}^{m \times n}$ (respectively, $\mathbb{C}^{m \times n}$) denotes the set of $m \times n$ real (respectively, complex) matrices. Furthermore, $\|\cdot\|$ denotes vector or matrix norm, $\|\cdot\|$ denotes vector or matrix operator norm, and $H_2^{m \times n}$ and $H_\infty^{m \times n}$ denote the sets of $m \times n$ analytic matrix functions $G : j\mathbb{R} \rightarrow \mathbb{C}^{m \times n}$ with norms $\|G\|_2 \triangleq \left[\int_{-\infty}^{\infty} \|G(j\omega)\|_F^2 d\omega \right]^{1/2}$ and $\|G\|_\infty \triangleq \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(j\omega))$, respectively, where $\|\cdot\|_F$ and $\sigma_{\max}(\cdot)$ denote the Frobenius norm and the maximum singular value, respectively. Finally, L_p^n denotes the set of n -dimensional vector

Manuscript received October 25, 2002; revised November 19, 2002. Recommended by Associate Editor Y. Wardi.

V. Chellaboina is with the Mechanical and Aerospace Engineering Department, the University of Missouri–Columbia, Columbia, MO 65211-2200 USA (e-mail: ChellaboinaV@missouri.edu).

W. M. Haddad is with the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150 USA (e-mail: wm.haddad@aerospace.gatech.edu).

Digital Object Identifier 10.1109/TAC.2003.809161

functions $w : \mathcal{J}\mathbb{R} \rightarrow \mathbb{C}^n$ with norm $\|w\|_p \triangleq \left[\int_{-\infty}^{\infty} \|w(j\omega)\|^p d\omega \right]^{1/p}$, where $p \in [1, \infty]$ and $\|\cdot\|$ is a vector norm defined on \mathbb{C}^n .

Let \mathcal{X} be a normed linear space over the field \mathbb{C} with norm $\|\cdot\|$ and let $\mathcal{B}(\mathcal{X})$ denote the space of bounded linear operators $T : \mathcal{X} \rightarrow \mathcal{X}$ with norm on $\mathcal{B}(\mathcal{X})$ defined by

$$\|T\| \triangleq \sup_{x \in \mathcal{X}, \|x\|=1} \|T(x)\|. \quad (1)$$

Note that for every $T \in \mathcal{B}(\mathcal{X})$, $\|T(x)\| \leq \|T\| \|x\|$ and $\|I\| = 1$, where $I : \mathcal{X} \rightarrow \mathcal{X}$ denotes the identity operator given by $I(x) = x$.

Theorem 1: $\|x\| = \sup_{T \in \mathcal{B}(\mathcal{X}), \|T\|=1} \|Tx\|$.

Proof: The result is immediate by noting that $\|T(x)\| \leq \|T\| \|x\|$ and $\|I\| = 1$. \square

Remark 1: Theorem 1 shows that every norm $\|\cdot\|$ on \mathcal{X} is induced by $\|\cdot\|'$ and $\|\cdot\|''$. In the case where $\mathcal{X} = \mathbb{C}^{m \times n}$, $\|\cdot\| = \|\cdot\|_F$, and $\|\cdot\|' = \|\cdot\|_F$, Theorem 1 shows that every matrix norm $\|\cdot\|$ on $\mathbb{C}^{m \times n}$ is induced by $\|\cdot\|'$ and $\|\cdot\|''$.

Let $\mathcal{X} = \mathbb{C}^{m \times n}$ and $\mathcal{B}(\mathcal{X}) = \mathbb{C}^{n \times n}$ so that for every $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times n}$ the linear operator $T \in \mathcal{B}(\mathcal{X})$ is given by $T(A) = AB \in \mathbb{C}^{m \times n}$. Now, if $\|\cdot\| = \|\cdot\|_F$, then it can be easily shown that $\|T\| = \sigma_{\max}(B)$ and hence it follows from Theorem 1 that $\|A\|_F = \max_{B \in \mathbb{C}^{n \times n}, \sigma_{\max}(B)=1} \|AB\|_F$ yielding [2, Prop. 1]. Alternatively, let $\mathcal{X} = H_2^{m \times n}$ and $\mathcal{B}(\mathcal{X}) = H_{\infty}^{n \times n}$ so that for every $G(\cdot) \in H_2^{m \times n}$ and $K(\cdot) \in H_{\infty}^{n \times n}$ the linear operator $T \in \mathcal{B}(\mathcal{X})$ is given by $T(G) = GK \in H_2^{m \times n}$. Now, if $\|\cdot\| = \|\cdot\|_2$, then

$$\begin{aligned} \|GK\|_2 &= \left[\int_{-\infty}^{\infty} \|G(j\omega)K(j\omega)\|_F^2 d\omega \right]^{1/2} \\ &\leq \left[\int_{-\infty}^{\infty} \sigma_{\max}^2(K(j\omega)) \|G(j\omega)\|_F^2 d\omega \right]^{1/2} \\ &\leq \|K\|_{\infty} \left[\int_{-\infty}^{\infty} \|G(j\omega)\|_F^2 d\omega \right]^{1/2} \\ &= \|G\|_2 \|K\|_{\infty}. \end{aligned} \quad (2)$$

Equality in (2) can be achieved by choosing $G(j\omega) = \delta^{1/2}(\omega - \hat{\omega})\hat{G}$, $\omega \in \mathbb{R}$, where $\delta : \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{R} \cup \{\infty\}$ is the Dirac delta function, $\hat{\omega} \in \mathbb{R} \cup \{\infty\}$ is such that $\sigma_{\max}(K(j\hat{\omega})) = \|K\|_{\infty}$, and $\hat{G} \in \mathbb{C}^{m \times n}$ is such that $\|\hat{G}K(j\hat{\omega})\|_F = \sigma_{\max}(K(j\hat{\omega}))\|\hat{G}\|_F$.¹ Thus, $\|K\|_{\infty} = \sup_{G(\cdot) \in H_2, \|G\|_2=1} \|GK\|_2$ and hence it follows from Theorem 1 that $\|G\|_2 = \sup_{K \in H_{\infty}^{n \times n}, \|K\|_{\infty}=1} \|GK\|_2$ yielding [2, Prop. 2].

Next, let \mathcal{X} and \mathcal{Y} be normed linear spaces with operator norms $\|\cdot\|$ and $\|\cdot\|''$, respectively. Now, let $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ denote the space of bounded linear operators $T : \mathcal{X} \rightarrow \mathcal{Y}$ with norm on $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ defined by

$$\|T\|' \triangleq \sup_{x \in \mathcal{X}, \|x\|=1} \|T(x)\|''. \quad (3)$$

Once again for every $x \in \mathcal{X}$ and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, $\|T(x)\|'' \leq \|T\|' \|x\|$. However, in this case an identity operator $I : \mathcal{X} \rightarrow \mathcal{Y}$ exists if and only if $\mathcal{X} \subset \mathcal{Y}$. Furthermore, $\|I\|' = 1$ only if $\|x\|'' \leq \|x\|$, $x \in \mathcal{X}$. Hence, the equality

$$\|x\| = \sup_{T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}), \|T\|'=1} \|Tx\|'' \quad (4)$$

may not always hold. If (4) holds, then it follows that $\|\cdot\|$ is induced by $\|\cdot\|'$ and $\|\cdot\|''$. The following problem regarding bounded linear operators captures the essence of the problem addressed in [1].

Problem 1: For a given pair of normed linear spaces \mathcal{X} and \mathcal{Y} with norms $\|\cdot\|$ and $\|\cdot\|''$, respectively, determine conditions under which (4) holds. More generally, given a normed linear space \mathcal{X} with norm $\|\cdot\|$ and a vector space \mathcal{Y} determine conditions under which there exists a norm $\|\cdot\|''$ defined on \mathcal{Y} such that (4) holds.

¹For details of a similar proof, see [3, p. 677].

In the case where $\mathcal{X} = \mathbb{C}^{m \times n}$, $\mathcal{Y} = \mathbb{C}^m$, and $\|\cdot\| = \|\cdot\|_F$, we showed in [1] that there does not exist $\|\cdot\|''$ on \mathbb{C}^m such that (4) holds. Finally, in the case where $\mathcal{X} = H_2^{m \times n}$ and $\|\cdot\| = \|\cdot\|_2$, Problem 1 is directly related to the question regarding whether (or not) the H_2 norm is induced on particular *vector-valued* operator spaces [1]. Specifically, do there exist L_p^n and L_q^m spaces such that for every $G \in H_2^{m \times n}$, $G : L_p^n \rightarrow L_q^m$ and

$$\|G\|_2 = \sup_{w \in L_p^n, \|w\|_p=1} \|Gw\|_q.$$

This is the relevant question in system theory since input and output spaces are typically vector operators and *not* matrix operators as considered in [2]. This remains an open problem.

REFERENCES

- [1] V. Chellaboina and W. M. Haddad, "Is the Frobenius matrix norm induced?," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 2137–2139, 1995.
- [2] S. M. Djouadi, "Comments on 'Is the Frobenius matrix norm induced?'," *IEEE Trans. Automat. Contr.*, vol. 48, pp. 518–519, Mar. 2003.
- [3] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice-Hall, 2002.

Comments and Further Results on "A Descriptor System Approach to H_{∞} Control of Linear Time-Delay Systems"

Huijun Gao and Changhong Wang

Abstract—This note presents some comments and further results concerning the descriptor system approach to H_{∞} control of linear time-delay systems. Upon the system model of the paper by Fridman and Shaked, we propose a new iterative state-feedback controller design procedure, which is based on a new bounded real lemma derived upon an inequality recently proposed by Moon. The proposed design solves both the instantaneous and delayed feedback problems in a unified framework, and is illustrated by a numerical example to be much less conservative than the above paper and other relevant references.

Index Terms—Delay-dependent performance, descriptor systems, H_{∞} control, linear matrix inequality (LMI), neutral delay systems.

I. INTRODUCTION AND COMMENTS

Recently, a new descriptor system approach was proposed for time-delayed systems, which significantly reduced the overdesign over the traditional methods due to the facts that it was based on a transformed model that was equivalent to the original system and fewer terms needed to be bounded in the derivation [1]–[6]. This approach has been exploited for different problems (see, e.g., [12] for neutral delay systems, [1] for distributed delay systems, [2] for positive realness problems, and [3] for filtering problems), and has been gradually improved by seeking different bounding techniques to derive the bounds on the inner product of two vectors (see, e.g., standard bounding technique was used in [1], [3]–[5], the inequality

Manuscript received July 8, 2002; revised October 10, 2002. Recommended by Associate Editor L. Pandolfi. This work was supported in part by the National Natural Science Foundation of China under Grant 69874008.

The authors are with the Inertial Navigation Center, Harbin Institute of Technology, 150001 Harbin, China (e-mail: hjgao@hotmail.com).

Digital Object Identifier 10.1109/TAC.2003.809154