Fixed-Order Dynamic Compensation for Axial Flow Compression Systems

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***Abstract*—In this paper, we develop linear fixed-order (i.e., full- and reduced order) pressure rise feedback dynamic compensators for axial flow compressors with throttle valve actuation. Unlike the nonlinear static controllers proposed in the literature possessing gain at all frequencies, the proposed dynamic compensators explicitly account for compressor performance versus sensor accuracy, compressor performance versus processor throughput, and compressor performance versus disturbance rejection. Furthermore, the proposed controller is predicated on *only* pressure rise measurements, providing a considerable simplification in the sensing architecture over the bifurcation-based and backstepping controllers proposed in the literature.**

***Index Terms*—Axial flow compressors, disturbance rejection, pressure rise feedback, reduced order control, rotating stall, surge.**

I. Introduction

A FUNDAMENTAL development in compression system modeling for low-speed axial compressors is the Moore–Greitzer model given in [1]. Using the Moore–Greitzer model, a bifurcation-based controller for rotating stall was developed by Liaw and Abed [2]. In particular, the Liaw and Abed static nonlinear controller is given by

$$\gamma_{\rm throt}(A) = \gamma_0 + kA^2 \tag{1}$$

where $\gamma_{\rm throt}(A)$ is the control throttle, A is the amplitude of the rotating stall, and γ_0 and k are constants. However, as noted by Eveker *et al.* [3], even though the Liaw and Abed controller reduces the abrupt transition into rotating stall, it is ineffective for surge. Modifying the static nonlinear controller given by (1) to

$$\gamma_{\rm throt}(A, \dot{\Phi}) = \gamma_0 + k_1A^2 - k_2\dot{\Phi} \tag{2}$$

where $\dot{\Phi}$ is the time rate of change of the mean flow in the compressor[1] and k_1 and k_2 are constants, Badmus *et al.* [4] considerably extended the domain of attraction of the Liaw and Abed controller. A fundamental shortcoming of the aforementioned controllers is the demanding two-dimensional sensing requirements for implementing these controllers. Specifically, measuring rotating stall amplitude is quite challenging, requiring circumferentially distributed sensor arrays around the compressor annulus with discrete Fourier transform software for spatial and temporal filtering for computing the first circumferential spatial harmonic of rotating stall. As an alternative to the locally stabilizing nonlinear controllers developed in [2]–[4], the authors in [5]–[7] develop globally stabilizing controllers for controlling rotating stall and surge in axial flow compression systems. In particular, Lyapunov-based recursive backstepping globally stabilizing static *full-state* feedback nonlinear controllers requiring rotating stall amplitude measurements are constructed in [7], while a globally stabilizing static *output* feedback nonlinear controller is given in [6]. Specifically, the Krstić *et al.* [6] static output feedback controller is given by

$$\gamma_{\rm throt}(\Phi, \Psi) = \frac{\Gamma + c_1\Psi - c_2\Phi}{\sqrt{\Psi}} \tag{3}$$

where Ψ is the pressure rise in the compressor, Φ is the circumferencially averaged flow in the compressor,[2] and $\Gamma, c_1,$ and c_2 are constants. Even though (3) provides a simplified sensing architecture over (1) and (2), the controller is static, possessing gain at all frequencies. Furthermore, none of the above controllers have any disturbance rejection guarantees.

In this paper, we develop *linear* time-invariant *pressure rise* feedback reduced-order dynamic compensators for the nonlinear Moore–Greitzer axial flow compressor model. Specifically, we construct a modified Riccati equation whose solution guarantees that the nonlinear closed-loop axial flow compression system is locally asymptotically stable and the closed-loop output system energy is less than the net weighted input energy at any time T in the face of $\mathcal{L}_2$ exogenous disturbances. Using the modified Riccati equation, constructive sufficient conditions for fixed-order (i.e., full- and reduced-order) pressure rise feedback dynamic compensators

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[1]Even though a patented differentiation scheme for sensing the time rate of change of the mean flow in the compressor is given in [3], the calculation of $\dot{\Phi}$ can be simply obtained by the most elementary equations of fluid dynamics. For example, under the assumption of one-dimensional flow, the unsteady axial momentum equation as applied to the bulk of the fluid in the inlet duct yields $\dot{\Phi} = -(1/\rho LU)\Delta P$, where ρ is the fluid density, L is the duct length, U is the rotor wheel speed, and ΔP is the change in static pressure.

[2]Mean flow is relatively simple to measure and is usually measured using pitot probes located in the bell mouth of an engine.

guaranteeing local asymptotic stability and disturbance rejection are developed. Unlike the nonlinear static and relative degree zero controllers possessing gain at all frequencies discussed above, the proposed linear dynamic compensators explicitly account for compressor performance versus sensor accuracy, compressor performance versus processor throughput, and compressor performance versus disturbance rejection. Furthermore, the proposed controller is predicated on *only* pressure rise measurements, providing a considerable simplification in the sensing architecture.

II. The Output Feedback Disturbance Rejection Control Problem for Axial Flow Compression Systems

To capture post-stall transients in axial flow compression systems, we use the one-mode Galerkin approximation model for the partial differential equation characterizing the disturbance velocity potential at the compressor inlet proposed by Moore and Greitzer [1]. This model, without external disturbances, is given in [1], and with external disturbances is given by

$$\dot{A}(t) = \frac{\sigma}{2}A(t)\left[1-\Phi^2(t)-\frac{1}{4}A^2(t)\right]+\beta_1 w_1(t)$$
$$A(0)=A_0,\quad t\geq 0 \tag{4}$$
$$\dot{\Phi}(t) = -\Psi(t)+\Psi_C(\Phi(t))-\frac{3}{4}\Phi(t)A^2(t)+\beta_2 w_2(t)$$
$$\Phi(0)=\Phi_0 \tag{5}$$
$$\dot{\Psi}(t)=\frac{1}{\beta^2}[\Phi(t)-\Phi_T(t)]+\beta_3 w_3(t)$$
$$\Psi(0)=\Psi_0 \tag{6}$$

where Φ is the circumferentially averaged axial mass flow in the compressor, Ψ is the total-to-static pressure rise, A is the normalized stall cell amplitude of angular variation capturing a measure of nonuniformity in the flow, Φ_T is the mass flow through the throttle, σ and β are positive constant parameters, $\Psi_C(\cdot)$ is a given compressor pressure-flow map, and $w_1(t), w_2(t)$ and $w_3(t), t\geq 0$ are $\mathcal{L}_2$ external disturbance signals with scaling factors $\beta_1,\beta_2,\beta_3 \in \mathbb{R}$. The specific compressor pressure-flow map $\Psi_C(\cdot)$, which was considered in [1], is $\Psi_C(\Phi)=\Psi_{C_0}+1+(3/2)\Phi-(1/2)\Phi^3$, where Ψ_{C_0} is a constant parameter. Furthermore, we consider a throttle characteristic given by $\Psi_T = -(1/\gamma_{\rm throt}^2)\Phi_T^2$, where Ψ_T is the drop in pressure through the throttle. The proposed additive disturbance model can be used to capture combustion noise and turbine speed fluctuations. For example, $w_3(t), t\geq 0$, might reflect back-pressure disturbances to the compressor from the combustor.

Next, note that for fixed values of flow through the throttle, $\Phi_T(t)\equiv\Phi_{T_{\rm eq}}$, (4)–(6) have an equilibrium point given by $(A_{\rm eq},\Phi_{\rm eq},\Psi_{\rm eq}) = (0,\Phi_{T_{\rm eq}},\Psi_C(\Phi_{\rm eq}))$. Defining the shifted state variables $x_1 \triangleq A, x_2\triangleq \Phi-\Phi_{\rm eq}$, and $x_3\triangleq\Psi-\Psi_{\rm eq}$, so that for a given equilibrium point on the axisymmetric branch of the compressor characteristic pressure-flow map the system equilibrium is translated to the origin, it follows that the translated nonlinear system is given by

$$\dot{x}_1(t)=\frac{\sigma}{2}(1-\lambda^2)x_1(t)-\frac{\sigma}{2}\left[\frac{1}{4}x_1^3(t) + x_1(t)x_2^2(t)+2\lambda x_1(t)x_2(t)\right]+\beta_1 w_1(t)$$
$$x_1(0)=x_{1_0},\quad t\geq 0 \tag{7}$$
$$\dot{x}_2(t)=-x_3(t)+\frac{3}{2}(1-\lambda^2)x_2(t)-\frac{1}{2}x_2^3(t) - \frac{3}{2}\lambda x_2^2(t)-\frac{3}{4}\lambda x_1^2(t)-\frac{3}{4}x_1^2(t)x_2(t)+\beta_2 w_2(t)$$
$$x_2(0)=x_{2_0} \tag{8}$$
$$\dot{x}_3(t)=\frac{1}{\beta^2}[x_2(t)-u(t)]+\beta_3 w_3(t)$$
$$x_3(0)=x_{3_0} \tag{9}$$

where $\lambda\triangleq\Phi_{T_{\rm eq}}$ and $u\triangleq\Phi_T-\lambda$. Decomposing (7)–(9) into a linear and a nonlinear part, we obtain the state-space model

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}\frac{\sigma}{2}(1-\lambda^2) & 0 & 0\\ 0 & \frac{3}{2}(1-\lambda^2) & -1\\ 0 & \frac{1}{\beta^2} & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}-\frac{\sigma}{8} & 0\\ 0 & -\frac{1}{4}\\ 0 & 0\end{bmatrix}\phi(y_0(t)) + \begin{bmatrix}0\\ 0\\ -\frac{1}{\beta^2}\end{bmatrix}u(t) + \begin{bmatrix}\beta_1 & 0 & 0\\ 0 & \beta_2 & 0\\ 0 & 0 & \beta_3\end{bmatrix}\begin{bmatrix}w_1(t)\\ w_2(t)\\ w_3(t)\end{bmatrix} \tag{10}$$

where

$$y_0(t)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$
$$\phi(y_0(t))=\begin{bmatrix}x_1^3(t)+4x_1(t)x_2^2(t)+8\lambda x_1(t)x_2(t)\\ 3\lambda x_1^2(t)+3x_1^2(t)x_2(t)+6\lambda x_2^2(t)+2x_2^3(t)\end{bmatrix}.$$

Now, it can be shown that the linear part of (10) is linearly stabilizable for $\lambda>1$, while for $\lambda=1$, corresponding to the maximum pressure rise equilibrium point, the linear part of (10) is linearly unstabilizable. With the system written in this form, we can now state the dynamic output feedback control problem for axial flow compression systems. For generality of exposition, we first present the formulation for an n-dimensional nonlinear dynamical system and then specialize it to the system given by (10).

Dynamic Output Feedback Control Problem for Nonlinear Systems: Given the nth-order stabilizable and detectable[3] nonlinear dynamical system

$$\dot{x}(t)=Ax(t)+B_0\phi(y_0(t))+Bu(t)+D_1w(t)$$
$$x(0)=x_0,\qquad t\geq 0 \tag{11}$$
$$y_0(t)=C_0x(t) \tag{12}$$

[3]Here, stabilizability and detectability are defined with respect to the linear part of the dynamical system (11), (12).

with output measurements $y(t) = Cx(t) + D_2w(t)$, where $u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^l, y_0(t) \in \mathbb{R}^{l_0}, t \geq 0, \phi : \mathbb{R}^{l_0} \to \mathbb{R}^{m_0}$, and where $w(t) \in \mathbb{R}^d, t \geq 0$, is an exogenous $\mathcal{L}_2$ signal, each of whose components has norm less than one, determine an n_cth-order linear time-invariant dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \qquad x_c(0) = x_{c_0}, \qquad t \geq 0 \quad (13)$$
$$u(t) = C_c x_c(t) \quad (14)$$

that satisfies the following design criteria.

1) The undisturbed $(w(t) \equiv 0)$ closed-loop nonlinear system (11)–(14) is asymptotically stable.
2) The disturbed closed-loop system (11)–(14) from $\mathcal{L}_2$ disturbances $w(\cdot)$ to performance variables $z(t) = E_1x(t) + E_2u(t)$, satisfies the disturbance rejection constraint

$$\int_0^T z^T(s)z(s)\,ds < \gamma_d^2 \int_0^T w^T(s)w(s)\,ds + V(\tilde{x}(0)), \quad T \geq 0, \qquad w(\cdot) \in \mathcal{L}_2 \quad (15)$$

where $z(t) \in \mathbb{R}^p, t \geq 0, \gamma_d > 0$ is a given constant, $\tilde{x}(t) \triangleq [x^T(t) \quad x_c^T(t)]^T$, and $V(\cdot)$ is a Lyapunov function for the closed-loop system (11)–(14).
3) The quadratic performance functional

$$J(A_c, B_c, C_c) \triangleq \int_0^\infty z^T(t)z(t)\,dt \quad (16)$$

with $w(t) \equiv 0$ is minimized.

Note that sensor accuracy can be accounted for through the matrix D_2. Furthermore, processor throughput is enforced by setting the order of the linear compensator given by (22) and (23). For the axial compressor model given by (10), the plant is only third order, and thus reduced order control is not necessary. For higher order models, however, the ability to use reduced order control becomes much more important. Finally, disturbance rejection levels are set with the parameter γ_d.

We now specialize this problem to the axial compressor. Specifically, for the three-state parameterized Moore–Greitzer model given by (10) with pressure rise measurements, the system matrices in (11) and (12) are given by

$$A = \begin{bmatrix} \frac{\sigma}{2}(1-\lambda^2) & 0 & 0 \\ 0 & \frac{3}{2}(1-\lambda^2) & -1 \\ 0 & \frac{1}{\beta^2} & 0 \end{bmatrix}$$
$$B_0 = \begin{bmatrix} -\frac{\sigma}{8} & 0 \\ 0 & -\frac{1}{4} \\ 0 & 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (17)$$
$$B = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{\beta^2} \end{bmatrix}, \quad D_1 = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix}, \quad C = [0 \quad 0 \quad 1]. \quad (18)$$

III. Sufficient Conditions for Closed-Loop Stability and Disturbance Rejection

In this section, we provide a Riccati equation that guarantees asymptotic stability of the undisturbed $(w(t) \equiv 0)$ closed-loop system (11)–(14) as well as disturbance rejection of the disturbed closed-loop system in the face of $\mathcal{L}_2$ exogenous disturbances. First, however, note that the closed-loop system (11)–(14) has a state-space representation given by

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}_0\phi(y_0(t)) + \tilde{D}w(t)$$
$$\tilde{x}(0) = \tilde{x}_0, \quad t \geq 0 \quad (19)$$
$$z(t) = \tilde{E}\tilde{x}(t) \quad (20)$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}$$
$$\tilde{B}_0 \triangleq \begin{bmatrix} B_0 \\ 0_{n_c \times m_0} \end{bmatrix}, \quad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix}$$
$$y_0(t) = \tilde{C}_0\tilde{x}(t), \quad \tilde{C}_0 \triangleq [C_0 \quad 0_{l_0 \times n_c}], \quad \tilde{E} \triangleq [E_1 \quad E_2C_c].$$

Furthermore, we assume that the nonlinear part of (11), or, equivalently, (19), is such that $\phi(\cdot) \in \Phi_{\mathcal{D}}$, where

$$\Phi_{\mathcal{D}} \triangleq \{\phi : \mathcal{D} \to \mathbb{R}^{m_0} : \phi(0) = 0, \|\phi(y_0)\|_2^2 \leq \gamma_n^{-2}\|y_0\|_2^2, y_0 \in \mathcal{D}\} \quad (21)$$

where $\mathcal{D} \subseteq \mathbb{R}^{l_0}$ is a closed set and $\gamma_n > 0$ is given. For the statement of the main result of this section, define the notation $\tilde{n} \triangleq n + n_c, \tilde{R} \triangleq \tilde{E}^T\tilde{E}$, and $\tilde{V} \triangleq \tilde{D}\tilde{D}^T$ and set $\mathcal{D} = \mathbb{R}^{l_0}$.

Theorem 3.1: Let (A_c, B_c, C_c) be given and suppose there exists an $\tilde{n} \times \tilde{n}$ positive–definite matrix $\tilde{P}$ and scalars $\epsilon, \gamma_d, \gamma_n > 0$ satisfying

$$0 = \tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} + \gamma_d^{-2}\tilde{P}\tilde{V}\tilde{P} + \gamma_n^{-2}\tilde{P}\tilde{B}_0\tilde{B}_0^T\tilde{P} + \tilde{C}_0^T\tilde{C}_0 + \epsilon\tilde{P} + \tilde{R}. \quad (22)$$

Then the function $V(\tilde{x}) = \tilde{x}^T\tilde{P}\tilde{x}$, is a Lyapunov function that guarantees that the undisturbed $(w(t) \equiv 0)$ closed-loop system (19), (20) is globally asymptotically stable for all $\phi(\cdot) \in \Phi_{\mathbb{R}^{l_0}}$. Furthermore, the solution $\tilde{x}(t), t \geq 0$, of the nonlinear system (19), (20) satisfies the disturbance rejection constraint

$$\int_0^T z^T(s)z(s)\,ds < \gamma_d^2 \int_0^T w^T(s)w(s)\,ds + V(\tilde{x}_0) \quad T \geq 0, \qquad w(\cdot) \in \mathcal{L}_2. \quad (23)$$

Finally, in the case where $w(t) \equiv 0$, the performance functional (16) satisfies the bound

$$J(\tilde{x}_0, A_c, B_c, C_c) = \int_0^\infty z(t)^T z(t)\,dt < V(\tilde{x}_0). \quad (24)$$

Proof: First note that since $\tilde{P}$ is positive definite, it follows that the Lyapunov function candidate $V(\tilde{x}) = \tilde{x}^T\tilde{P}\tilde{x}$ is positive definite for $\tilde{x} \neq 0$. The corresponding Lyapunov derivative along the trajectories $\tilde{x}(t), t \geq 0$, of the undisturbed $(w(t) \equiv 0)$ closed-loop system (19), (20) is given by

$$\begin{aligned}\dot{V}(\tilde{x}(t)) &\triangleq V'(\tilde{x}(t))[\tilde{A}\tilde{x}(t) + \tilde{B}_0\phi(y_0(t))] \\ &= \tilde{x}^T(t)(\tilde{A}^T\tilde{P} + \tilde{P}\tilde{A})\tilde{x}(t) \\ &\quad + 2\tilde{x}(t)^T\tilde{P}\tilde{B}_0\phi(y_0(t)), \qquad t \geq 0 \end{aligned} \tag{25}$$

or, equivalently, using (22)

$$\begin{aligned}\dot{V}(\tilde{x}(t)) = &-\tilde{x}^T(t)\left[\epsilon\tilde{P} + \gamma_d^{-2}\tilde{P}\tilde{V}\tilde{P} + \tilde{R}\right]\tilde{x}(t) \\ &- \gamma_n^{-2}\tilde{x}^T(t)\tilde{P}\tilde{B}_0\tilde{B}_0^T\tilde{P}\tilde{x}(t) \\ &- \tilde{x}^T(t)\tilde{C}_0^T\tilde{C}_0\tilde{x}(t) + \phi^T(y_0(t))\tilde{B}_0^T\tilde{P}\tilde{x}(t) \\ &+ \tilde{x}^T(t)\tilde{P}\tilde{B}_0\phi(y_0(t)), \qquad t \geq 0. \end{aligned} \tag{26}$$

Now, adding and subtracting $\gamma_n^2\phi^T(y_0(t))\phi(y_0(t)), t \geq 0$, to and from (26) and grouping terms yields

$$\begin{aligned}\dot{V}(\tilde{x}(t)) = &-\tilde{x}^T(t)\left[\epsilon\tilde{P} + \gamma_d^{-2}\tilde{P}\tilde{V}\tilde{P} + \tilde{R}\right]\tilde{x}(t) \\ &+ \gamma_n^2\phi^T(y_0(t))\phi(y_0(t)) - y_0^T(t)y_0(t) \\ &- \left[\gamma_n^{-1}\tilde{B}_0^T\tilde{P}\tilde{x}(t) - \gamma_n\phi(y_0(t))\right]^T \\ &\times \left[\gamma_n^{-1}\tilde{B}_0^T\tilde{P}\tilde{x}(t) - \gamma_n\phi(y_0(t))\right], \qquad t \geq 0. \end{aligned} \tag{27}$$

Since $\epsilon\tilde{P}$ is positive definite, $\gamma_d^{-2}\tilde{P}\tilde{V}\tilde{P} + \tilde{R}$ is nonnegative definite, and $\gamma_n^2\phi^T\phi - y_0^Ty_0 \leq 0$, it follows that $\dot{V}(\tilde{x}(t)) < 0, t \geq 0$, and hence the undisturbed $(w(t) \equiv 0)$ nonlinear closed-loop system (19), (20) is globally asymptotically stable.

Next, to show that the disturbance rejection constraint (23) holds, note that

$$\begin{aligned}0 &\leq \left(\gamma_d^{-1}\tilde{D}^T\tilde{P}\tilde{x} - \gamma_d w\right)^T\left(\gamma_d^{-1}\tilde{D}^T\tilde{P}\tilde{x} - \gamma_d w\right) \\ &= \gamma_d^{-2}\tilde{x}^T\tilde{P}\tilde{V}\tilde{P}\tilde{x} - 2\tilde{x}^T\tilde{P}\tilde{D}w + \gamma_d^2 w^T w \\ &\quad + \tilde{x}^T\tilde{R}\tilde{x} - z^T z, \qquad w \in \mathbb{R}^d. \end{aligned} \tag{28}$$

Now, let $w(\cdot) \in \mathcal{L}_2$ and let $\tilde{x}(t), t \geq 0$, denote the solution of the nonlinear closed-loop system (19), (20). Then

$$\begin{aligned}\dot{V}(\tilde{x}(t)) = \tilde{x}^T(t)(\tilde{A}^T\tilde{P} + \tilde{P}\tilde{A})\tilde{x}(t) - 2\phi^T(y_0(t))\tilde{B}_0^T\tilde{P}\tilde{x}(t) \\ + 2\tilde{x}^T(t)\tilde{P}\tilde{D}w(t), \quad t \geq 0 \end{aligned} \tag{29}$$

which, using (22) and (28), implies

$$\begin{aligned}\dot{V}(\tilde{x}(t)) &< 2\tilde{x}^T(t)\tilde{P}\tilde{D}w(t) - \gamma_d^{-2}\tilde{x}^T(t)\tilde{P}\tilde{V}\tilde{P}\tilde{x}(t) \\ &\quad - \tilde{x}^T(t)\tilde{R}\tilde{x}(t) \\ &\leq \gamma_d^2 w^T(t)w(t) - z^T(t)z(t), \qquad t \geq 0. \end{aligned} \tag{30}$$

Now, integrating (30) over $[0, T]$ yields

$$\begin{aligned}V(\tilde{x}(T)) < \int_0^T \left[\gamma_d^2 w^T(s)w(s) - z^T(s)z(s)\right] ds + V(\tilde{x}(0)) \\ T \geq 0, \qquad w(\cdot) \in \mathcal{L}_2 \end{aligned} \tag{31}$$

which, by noting that $V(\tilde{x}(T)) > 0, T \geq 0$, yields (23).

Finally, to show that the performance functional (16) satisfies the bound (24), note that (27) implies $\dot{V}(\tilde{x}(t)) < \tilde{x}^T(t)\tilde{R}\tilde{x}(t)$. Now, integrating over $[0, \infty)$ yields

$$\int_0^\infty z^T(t)z(t)\,dt < -\int_0^\infty \dot{V}(\tilde{x}(t))\,dt. \tag{32}$$

Next, since $\tilde{x}(t) \to 0$ as $t \to \infty$, where $\tilde{x}(t), t \geq 0$, satisfies (19) with $w(t) \equiv 0$, we obtain

$$\begin{aligned}J(\tilde{x}_0, A_c, B_c, C_c) &< V(\tilde{x}(0)) - \lim_{t\to\infty} V(\tilde{x}(t)) \\ &= V(\tilde{x}(0)) = \tilde{x}_0^T\tilde{P}\tilde{x}_0. \end{aligned}$$

□

Theorem 3.1 guarantees global asymptotic stability of a general nonlinear dynamical system if $\phi(\cdot) \in \Phi_{\mathcal{D}}$ with $\mathcal{D} = \mathbb{R}^{l_0}$. However, for the three-state axial compressor model given by (10), $\phi(\cdot) \in \Phi_{\mathcal{D}}$ is not satisfied for $\mathcal{D} = \mathbb{R}^{l_0}$ and global asymptotic stability cannot be guaranteed. Hence, to obtain a *local* stability result for the parameterized compressor model given by (10), we restrict $\mathcal{D}$ to the set $\mathcal{D}_c$, where $\mathcal{D}_c$ is the smallest compact set given by

$$\mathcal{D}_c \triangleq \left\{y_0 \in \mathbb{R}^{l_0} : \|\phi(y_0)\|_2^2 \leq \gamma_n^{-2}\|y_0\|_2^2\right\} \tag{33}$$

where $\phi(y_0), y_0 \in \mathbb{R}^{l_0}$, is the nonlinear part of (10).

Proposition 3.1: For the axial flow compression system given by (10), $\mathcal{D}_c$ is not empty.

Proof: Defining $f(y_0) \triangleq \gamma_n^{-2}y_0^Ty_0 - \phi^T(y_0)\phi(y_0)$, it follows that

$$\begin{aligned}f(y_0) = \gamma_n^{-2}\left(x_1^2 + x_2^2\right) - x_1^2\left[x_1^2 + 4x_2(2\lambda + x_2)\right]^2 \\ - \left[3x_1^2(\lambda + x_2) + 2x_2^2(3\lambda + x_2)\right]^2. \end{aligned} \tag{34}$$

Now, since $f'(0) = 0$, where $f'(0)$ denotes the Frechét derivative of $f(\cdot)$ at the origin, and

$$f''(0) = \begin{bmatrix} 2\gamma_n^{-2} & 0 \\ 0 & 2\gamma_n^{-2} \end{bmatrix} > 0 \tag{35}$$

where $f''(0)$ denotes the Hessian of $f(\cdot)$ at the origin, it follows that the origin is a local minimizer of $f(\cdot)$, with $f(0) = 0$. Thus, since $f(\cdot)$ is continuous, there exists a neighborhood of the origin where $f(y_0) > 0, y_0 \in \mathbb{R}^{l_0} \setminus \{0\}$, and hence $\mathcal{D}_c$ is not empty. □

Next, note that $J(\tilde{x}_0, A_c, B_c, C_c) < \tilde{x}_0^T\tilde{P}\tilde{x}_0 = \operatorname{tr}\tilde{P}\tilde{x}_0\tilde{x}_0^T$, which has the same form as the standard $\mathcal{H}_2$ cost appearing in standard LQG theory. Hence, we replace $\tilde{x}_0\tilde{x}_0^T$ by $\hat{V} \triangleq \operatorname{diag}[\hat{V}_1, B_c\hat{V}_2 B_c{}^T]$, where $\hat{V}_1 \in \mathbb{R}^{n\times n}$ and $\hat{V}_2 \in \mathbb{R}^{l\times l}$ are arbitrary design weights such that $\hat{V}_1 \geq 0$ and $\hat{V}_2 > 0$, and proceed by determining controller gains that minimize $\operatorname{tr}\tilde{P}\hat{V}$. Before proceeding, however, we shall require for technical reasons that $V_2 \triangleq D_2 D_2^T = \alpha^2\hat{V}_2$, where the positive scalar α is a design variable such that $\alpha \equiv 0$ if and only if $D_2 \equiv 0$. Next, in the spirit of [8], $\mathcal{J}(\tilde{P}, A_c, B_c, C_c) \triangleq \operatorname{tr}\tilde{P}\hat{V}$ can be interpreted as an auxiliary cost which leads to the following optimization problem.

Optimization Problem: Determine (A_c, B_c, C_c) that minimizes $\mathcal{J}(\tilde{P}, A_c, B_c, C_c) \triangleq \operatorname{tr} \tilde{P}\tilde{V}$ with $\tilde{P} > 0$ subject to (22) and such that (A_c, B_c, C_c) is minimal.

It follows from Theorem 3.1 that by deriving necessary conditions for the optimization problem, we obtain sufficient conditions for characterizing dynamic output feedback controllers guaranteeing closed-loop system stability and disturbance rejection to $\mathcal{L}_2$ exogenous disturbances.

IV. Reduced Order Dynamic Control for Axial Flow Compressors

In this section, we present our main theorem characterizing fixed-order disturbance rejection controllers for axial flow compression systems. For design flexibility, the compensator order, n_c, may be less than the plant order, n. For convenience in stating this result, define the notation $S \triangleq (I_n + \alpha^2\gamma_d^{-2}Q\hat{P})^{-1}$ for arbitrary nonnegative definite matrices $Q, \hat{P} \in \mathbb{R}^{n\times n}$, and $A_\epsilon \triangleq A + (1/2)\epsilon I_n, R_1 \triangleq E_1^T E_1, R_2 \triangleq E_2^T E_2, V_1 \triangleq D_1 D_1^T, \Sigma \triangleq BR_2^{-1}B^T$, and $\bar{\Sigma} \triangleq C^T \hat{V}_2^{-1} C$. Note that since Q and $\hat{P}$ are nonnegative definite and the eigenvalues of $Q\hat{P}$ coincide with the eigenvalues of the nonnegative definite matrix $Q^{1/2}\hat{P}Q^{1/2}$, it follows that $Q\hat{P}$ has nonnegative eigenvalues. Thus the eigenvalues of $I_n + \alpha^2\gamma_d^{-2}Q\hat{P}$ are all greater than one, so that S exists. Furthermore, define $\mathcal{D}_\beta \triangleq \{\tilde{x} \in \mathbb{R}^{\tilde{n}} : \tilde{C}_0\tilde{x} \in \mathcal{D}_c$ and $\tilde{x}^T\tilde{P}\tilde{x} \le \beta\}$, where $\tilde{P} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}, \tilde{P} > 0$, satisfies (22) for a given compensator (A_c, B_c, C_c), and $\beta > 0$.

Theorem 4.1: Let $n_c \le n$ and let $\epsilon, \gamma_n, \gamma_d, \alpha > 0$. Furthermore, suppose there exist $n \times n$ nonnegative-definite matrices $P, Q, \hat{P}$, and $\hat{Q}$ satisfying

$$0 = A_\epsilon^T P + PA_\epsilon + P\left[\gamma_n^{-2}B_0B_0^T + \gamma_d^{-2}V_1\right]P + R_1 + C_0^TC_0 - P\Sigma P + \tau_\perp^T P\Sigma P\tau_\perp, \tag{36}$$

$$0 = \left(A_\epsilon + \left[\gamma_n^{-2}B_0B_0^T + \gamma_d^{-2}V_1\right]\left[P + \hat{P}\right]\right)Q + Q\left(A_\epsilon + \left[\gamma_n^{-2}B_0B_0^T + \gamma_d^{-2}V_1\right]\left[P + \hat{P}\right]\right)^T + \hat{V}_1 - SQ\bar{\Sigma}QS^T + \tau_\perp SQ\bar{\Sigma}QS^T\tau_\perp^T \tag{37}$$

$$0 = \left(A_\epsilon - SQ\bar{\Sigma} + \left[\gamma_n^{-2}B_0B_0^T + \gamma_d^{-2}V_1\right]P\right)^T\hat{P} + \hat{P}\left(A_\epsilon - SQ\bar{\Sigma} + \left[\gamma_n^{-2}B_0B_0^T + \gamma_d^{-2}V_1\right]P\right) + \hat{P}\left(\gamma_n^{-2}B_0B_0^T + \gamma_d^{-2}\left[V_1 + \alpha^2 SQ\bar{\Sigma}QS^T\right]\right)\hat{P} + P\Sigma P - \tau_\perp^T P\Sigma P\tau_\perp, \tag{38}$$

$$0 = \left(A_\epsilon + \left[\gamma_n^{-2}B_0B_0^T + \gamma_d^{-2}V_1 - \Sigma\right]P\right)\hat{Q} + \hat{Q}\left(A_\epsilon + \left[\gamma_n^{-2}B_0B_0^T + \gamma_d^{-2}V_1 - \Sigma\right]P\right)^T + SQ\bar{\Sigma}QS^T - \tau_\perp SQ\bar{\Sigma}QS^T\tau_\perp^T \tag{39}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c \tag{40}$$

$$\hat{Q}\hat{P} = G^TM\Gamma, \qquad \Gamma G^T = I_{n_c}, \qquad M \in \mathbb{R}^{n_c\times n_c}$$
$$\tau \triangleq G^T\Gamma, \quad \tau_\perp \triangleq I_n - \tau \tag{41}$$

and let (A_c, B_c, C_c) be given by

$$A_c = \Gamma\left[A - SQ\bar{\Sigma} + \left(\gamma_n^{-2}B_0B_0^T + \gamma_d^{-2}V_1 - \Sigma\right)P\right]G^T \tag{42}$$

$$B_c = \Gamma SQC^T\hat{V}_2^{-1} \tag{43}$$

$$C_c = -R_2^{-1}B^TPG^T. \tag{44}$$

Then

$$\tilde{P} = \begin{bmatrix} P + \hat{P} & -\hat{P}G^T \\ -G\hat{P} & G\hat{P}G^T \end{bmatrix} \tag{45}$$

satisfies (22) and (A_c, B_c, C_c) is an extremal of $\mathcal{J}(\tilde{P}, A_c, B_c, C_c)$. Furthermore, the undisturbed $(w(t) \equiv 0)$ closed-loop system (19), (20) is globally asymptotically stable for all $\phi(\cdot) \in \Phi_{\mathbb{R}^{l_0}}$. Alternatively, if $\phi(\cdot) \in \Phi_{\mathcal{D}_c}$, then the undisturbed $(w(t) \equiv 0)$ closed-loop system (19)–(20) is locally asymptotically stable, and $\mathcal{D}_A$ defined by

$$\mathcal{D}_A \triangleq \{\tilde{x} \in \mathbb{R}^{\tilde{n}} : \tilde{C}_0\tilde{x} \in \mathcal{D}_c \text{ and } \tilde{x}^T\tilde{P}\tilde{x} \le v_\beta\} \tag{46}$$

where $v_\beta = \max\{\beta > 0 : \mathcal{D}_\beta \subset \mathbb{R}^{\tilde{n}}\}$, is a subset of the domain of attraction of the closed-loop system. Moreover, the solution $\tilde{x}(t), t \ge 0$, of the disturbed closed-loop system (19)–(20) satisfies the disturbance rejection constraint (23). Finally, the cost $\mathcal{J}(\tilde{P}, A_c, B_c, C_c)$ is given by $\mathcal{J}(\tilde{P}, A_c, B_c, C_c) = \operatorname{tr}[(P + \hat{P})\hat{V}_1 + \hat{P}SQ\bar{\Sigma}QS^T]$.

Proof: The proof is constructive in nature. For details of a similar proof, see [8] and [9]. □

In the full-order case, $n_c = n$, set $G = \Gamma = \tau = I_n$, so that $\tau_\perp = 0$. Now the last term in each of (36)–(39) can be deleted and G and Γ in (42)–(44) can be taken to be the identity. Furthermore, $\hat{Q}$ plays no role, so (39) is superfluous. If, alternatively, the reduced order constraint is retained and the disturbance rejection constraint is sufficiently relaxed, i.e., $\gamma_d \to \infty$, then considerable simplification arises in (36)–(39).

To solve the design equations (36)–(39), we employed the homotopic continuation method presented in [9]. Homotopy algorithms operate by first replacing the original problem with a simpler problem having a known solution. The desired solution is then reached by integrating along the homotopy path that connects the starting problem to the original problem. The algorithm was initialized with $\gamma_n = \gamma_d = \infty$, and the linear quadratic Gaussian (LQG) gains designed for the linear part of (10). For given values of the parameters γ_n and γ_d, the algorithm was used to find (A_c, B_c, C_c). After each iteration, γ_n and γ_d were decreased and the current values of (A_c, B_c, C_c) were used to find feasible values for γ_n and γ_d which were then used as the starting point for the next iteration. Complete details of a similar algorithm are provided in [9].

V. Active Dynamic Control of an Axial Flow Compressor

In this section, we use the design equations (36)–(39) to design a full-order $(n_c = 3)$ disturbance rejection, dynamic pressure rise feedback controller for the nonlinear Moore–Greitzer axial flow compressor model given in Section II. Specifically, we choose the data parameter values of $\sigma = 3.6, \beta = 0.356$, and $\Psi_{C_0} = 0.72$, and set $\lambda = 1.1$ in the parameterization given by (10). Note that with $\lambda = 1.1$, the linear part of (10) is linearly stabilizable, with (10) providing an equilibrium point close to the desired $(\lambda = 1.0)$ maximum pressure rise compressor operating point. Furthermore, we set $\gamma_n = 12.59$ and $\gamma_d = 2.3$, and choose design weights $V_1 = \hat{V}_1 = BB^T, V_2 = \hat{V}_2 = 1, E_1 = [10\ 1\ 1]$, and $E_2 = [0\ 1]^T$.

Using the initial conditions $A_0 = 1.0, \Phi_0 = 1.8$, and $\Psi_0 = 3.2$ to capture system transients in the compressor, simulations

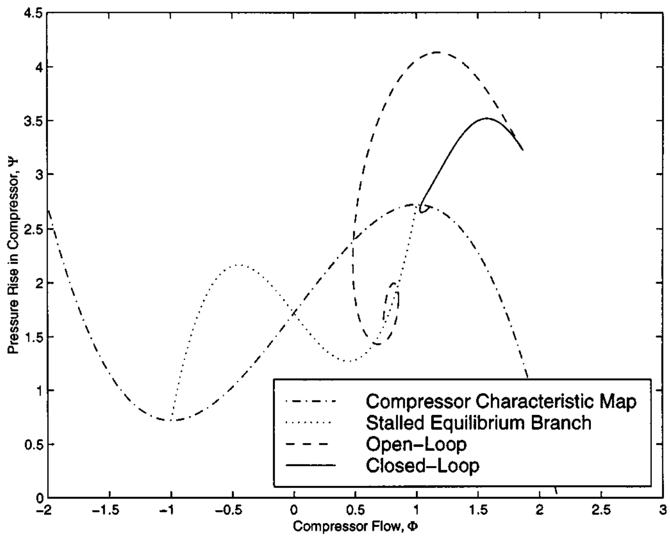

Fig. 1. Phase portrait of pressure versus flow.

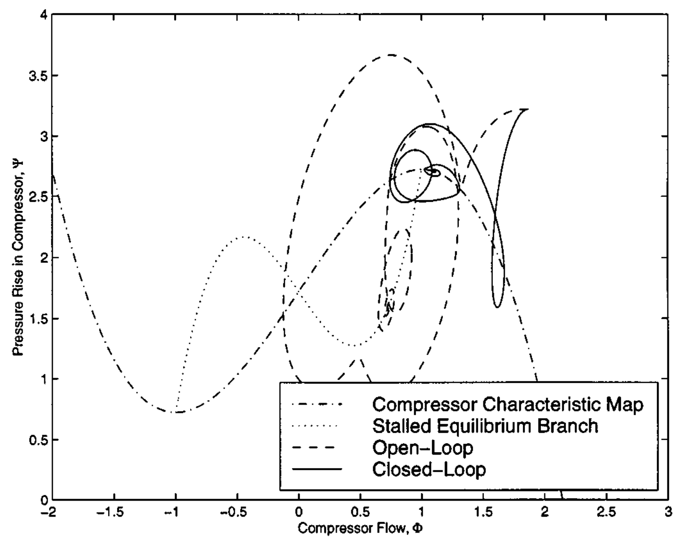

Fig. 3. Phase portrait of pressure versus flow with exogenous disturbances.

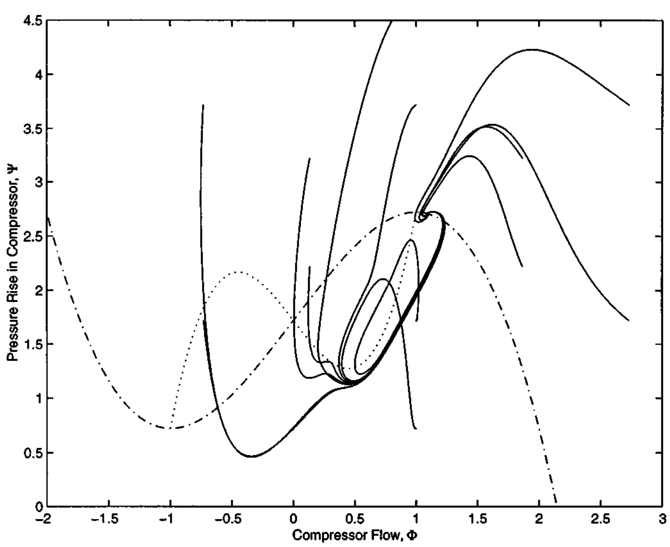

Fig. 2. Phase portrait of pressure versus flow of closed-loop system.

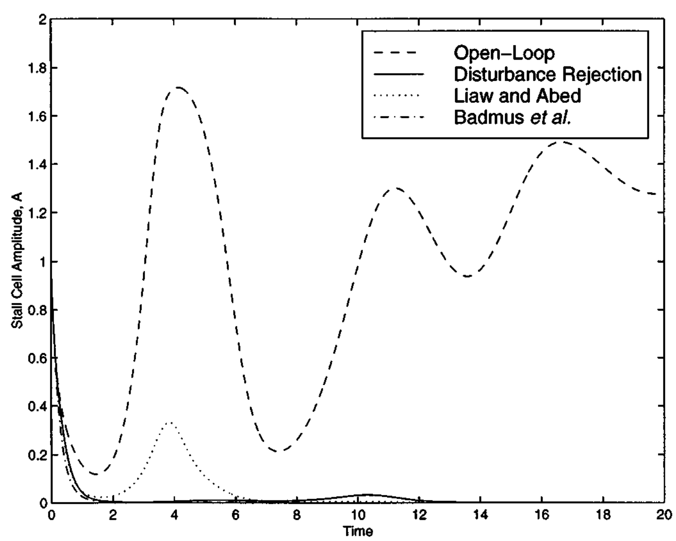

Fig. 4. Stall cell amplitude versus time.

of the Moore–Greitzer model were run in both the open-loop $(\gamma_{\text{throt}}(t) = \gamma_0 = 0.6689)$ configuration and with the designed controller given by (42)–(44) in the feedback loop. Fig. 1 shows the phase portrait of pressure rise versus flow in the compresssor for the case where no external disturbances are included in the simulation. It is seen from this plot that the controlled system converges to an equilibrium point close to the maximum pressure rise equilibrium point, whereas the constant throttle opening drives the system to a stalled equilibrium point. Fig. 2 shows the phase portrait of pressure rise versus flow for the closed-loop system at various initial conditions about the maximum pressure rise operating point. Note that the disturbance rejection controller was designed with $\gamma_n = 12.59$. This corresponds to a rather small guaranteed domain of attraction. However, the controller achieves a much larger domain of attraction, as shown by the distance from the initial conditions shown in Fig. 2 to the equilibrium point parameterized with $\lambda = 1.1$. Fig. 3 shows the phase portrait of pressure rise versus flow in the compressor when the exogenous $\mathcal{L}_2$ signal given by

$$w(t) = \begin{bmatrix} 0.9e^{-t/5}\sin(t+2) \\ 1.2e^{-t/10}\sin(t/2+1) \end{bmatrix}$$

is included in the simulations. Once again, the closed-loop system converges to a point near the maximum pressure rise equilibrium point while the constant throttle opening drives the system to a stalled equilibrium point.

Figs. 4–7 show the time histories of the stall cell amplitude $A(t), t \geq 0$, the compressor flow $\Phi(t), t \geq 0$, the pressure rise in the compressor $\Psi(t), t \geq 0$, and the control throttle opening $\gamma_{\text{throt}}(t), t \geq 0$, with the exogenous $\mathcal{L}_2$ disturbance included in the simulations for the constant throttle opening, the closed-loop system with the disturbance rejection controller (42)–(44), the Liaw and Abed controller [10] given by (1) with $k = 1$, and the Badmus *et al.* [4] controller given by (2) with $k_1 = 1$ and $k_2 = 1$. As seen in Fig. 4, the disturbance rejection controller

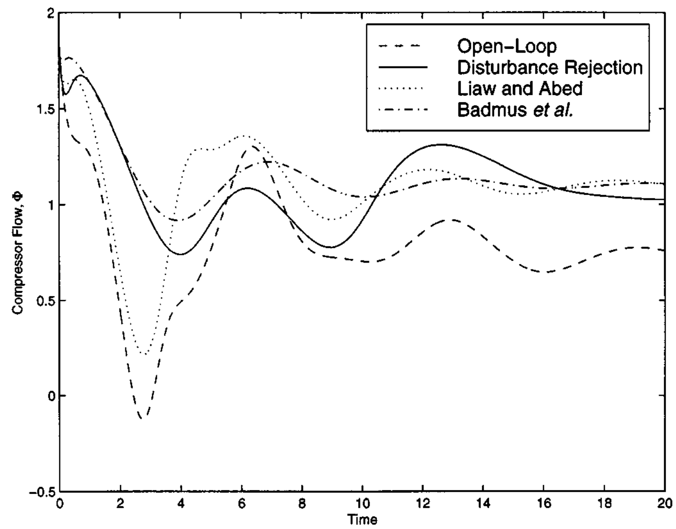

Fig. 5. Compressor flow versus time.

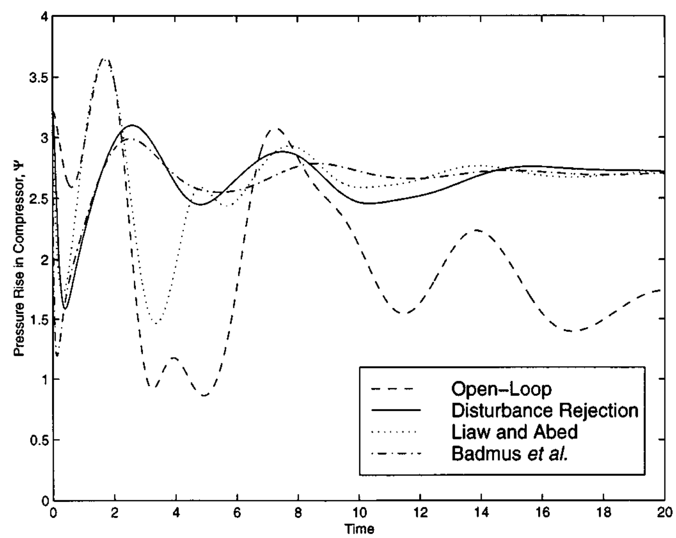

Fig. 6. Pressure rise in compressor versus time.

and the Badmus *et al.* controller reject the exogenous disturbance and drives the stall cell amplitude to zero, while the constant throttle opening is unable to reject the exogenous disturbance, driving the system to a stalled equilibrium. The Liaw and Abed [10] controller does stabilize the maximum pressure rise point but has poor disturbance rejection properties. Fig. 7 shows a comparison of the throttle opening for the three controllers considered as well as the constant throttle opening. It should be noted that the maximum throttle opening amplitude of the disturbance rejection controller is 1.1286, whereas the maximum throttle opening amplitude for the Liaw and Abed [10] controller is 1.669, and the Badmus *et al.* [4] controller is 5.018. Although input constraints would affect all the proposed control techniques, the Badmus *et al.* controller has a significantly larger throttle opening amplitude than the others. Even though the Badmus *et al.* controller exhibits very good system performance and disturbance rejection properties for the nominal system, in the presence of actuator constraints, which have not been modeled here, the Badmus *et al.* controller would be most

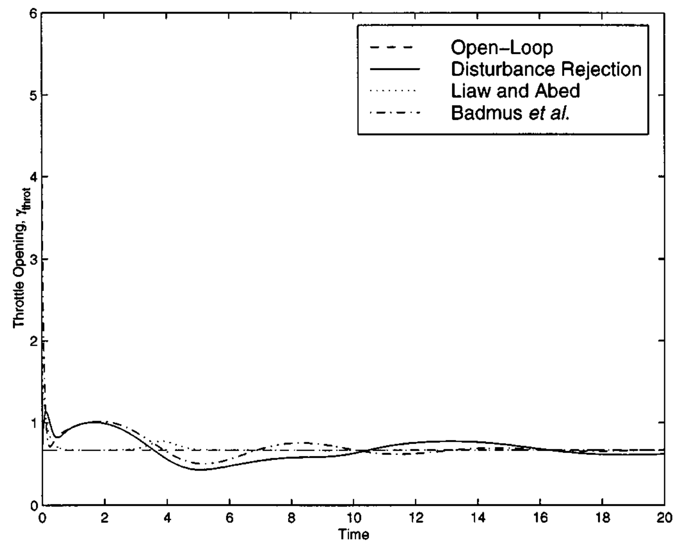

Fig. 7. Throttle opening versus time.

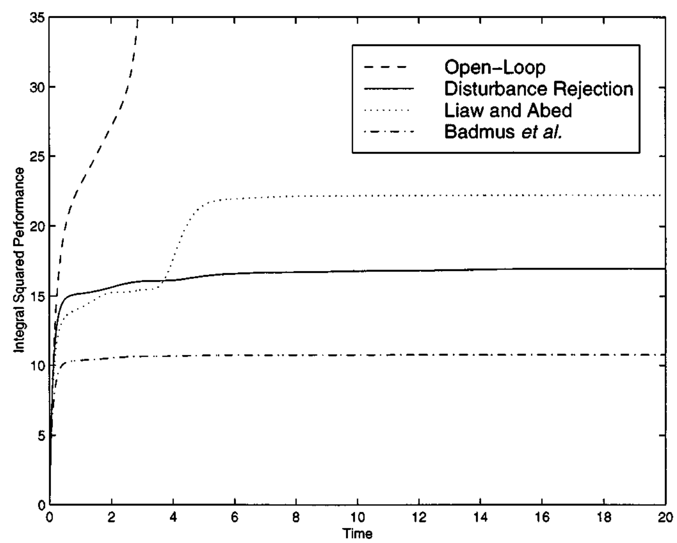

Fig. 8. Integral squared performance versus time.

affected, and its performance would degrade considerably. Finally, Fig. 8 shows the integral squared performance response $\int_0^T z^T(t)z(t)\,dt$ versus time for the three designs.

VI. CONCLUSION

A linear fixed-order (i.e., full- and reduced-order) pressure rise feedback dynamic compensation framework for axial flow compression systems was developed. Unlike the nonlinear bifurcation-based and backstepping controllers proposed in the literature, the proposed dynamic compensator framework explicitly accounts for compressor performance versus sensor noise, compressor performance versus controller order, and compressor performance versus disturbance rejection. Furthermore, the proposed pressure rise feedback controllers provide a considerable simplification in the sensing architecture required for controlling rotating stall and surge. Finally, we note that bifurcation-based controllers discussed in Section I, are dependent only upon measured quantities as opposed to the proposed controller which requires a model for the performance characteristic map. However, it is important to recognize that since the proposed controller guarantees *robust* stability for all $\phi(\cdot) \in \Phi_{\mathcal{D}_c}$, an accurate representation of the compressor characteristic map is *not* required as long as the compressor map is an element of $\Phi_{\mathcal{D}_c}$.

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