III. CONCLUSION

A special form of Kalman filter applicable to systems involving unknown biases and noise-free observations was derived. The optimal estimator was shown to involve a reduced-order filter for estimating the state, the order equalling the number of states less the number of noise-free measurements. This filtering arrangement offers in the reduced-order case the same advantages offered by the full-order separate-bias Kalman filter [1]—the potential for better numerical conditioning and reduced computational burden compared to that of the centralized Kalman filter based on state augmentation.

APPENDIX
REDUCED-ORDER KALMAN FILTER

The general reduced-order Kalman filter serves as a starting point for the derivation of the separate-bias form of the reduced-order Kalman filter. The specific form of the reduced-order Kalman filter used applies to the systems representable as

$$\dot{x} = Ax + Bu + F\xi \tag{39}$$

$$y = Cx \tag{40}$$

where $x \in \mathcal{R}^n$ is the state vector, $y \in \mathcal{R}^m$ is the observation vector, u is the control vector, and ξ is the white process noise vector with spectral density matrix Q. Observation noise is absent, as is the basic assumption with the reduced-order Kalman filter. It is also assumed, without any great loss in generality, that the state variables are defined so that the first m of them are measured directly (i.e., $C = [I \;\; 0]$) and the remaining $n - m$ are not measured at all. This corresponds to a partitioning of the state vector and matrices in (39) and (40) as follows:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u + \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \end{bmatrix} \xi. \tag{41}$$

(The overbars are used here for consistency with the notation employed in Section II.) As shown in [9], the reduced-order Kalman filter for the process with the matrices partitioned as above is given by

$$\hat{\bar{x}}_1 = y \tag{42}$$

$$\hat{\bar{x}}_2 = z + Ky \tag{43}$$

with

$$\dot{z} = (\bar{A}_{22} - K\bar{A}_{12})\hat{\bar{x}}_2 + (\bar{A}_{21} - K\bar{A}_{11} - \dot{K})y + (\bar{B}_2 - K\bar{B}_1)u. \tag{44}$$

The Kalman gain K and covariance P of the error in estimating $\bar{x}_2$ are given by

$$K = (P\bar{A}'_{12} + \bar{F}_2 Q \bar{F}'_1)W^{-1} \tag{45}$$

$$\dot{P} = \tilde{A}P + P\tilde{A}' - P\bar{A}'_{12}W^{-1}\bar{A}_{12}P + \bar{F}_2\tilde{Q}\bar{F}'_2 \tag{46}$$

where

$$\tilde{A} = \bar{A}_{22} - \bar{F}_2 Q \bar{F}'_1 W^{-1} \bar{A}_{12} \tag{47}$$

$$\tilde{Q} = Q - Q\bar{F}'_1 W^{-1} \bar{F}_1 Q$$

$$W = \bar{F}_1 Q \bar{F}'_1. \tag{48}$$

The time derivative of the Kalman gain matrix in (44) can be generated by differentiating (45) with the help of (46).

In these expressions it is assumed that the matrix W is nonsingular, or equivalently, that the submatrix $\bar{F}_1$ is of full rank. Thus, reduced-order Kalman filters of this form exist only for systems which have an independent source of noise driving each element of the vector of directly measured states $\bar{x}_1$ [9].

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Robust Stabilization for Continuous-Time Systems with Slowly Time-Varying Uncertain Real Parameters

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***Abstract*—In this paper the authors construct a new class of parameter-dependent Lyapunov functions to guarantee robust stability in the presence of time-varying rate-restricted plant uncertainty. Extensions to a class of time-varying nonlinear uncertainty that generalize the multivariable Popov criterion are also considered. These results are then used for controller synthesis to address the problem of robust stabilization in the presence of slowly time-varying real parameters.**

***Index Terms*—Absolute stability, Popov criterion, real parameter uncertainty, robust stabilization, time-varying uncertainty.**

I. INTRODUCTION

In a recent paper [5] a refined Lyapunov function technique was developed to overcome some of the current limitations of Lyapunov function theory for the problem of robust stability and performance

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in the presence of constant real parameter uncertainty. Specifically, a general framework for robust controller analysis and synthesis based on *parameter-dependent Lyapunov functions* was developed that is both flexible in addressing a large class of uncertainty structures and restrictive in excluding uncertainties that are not physically meaningful. The idea behind this framework is to allow the Lyapunov function to be a function of the uncertainty, thus guaranteeing robust stability and performance via a *family* of Lyapunov functions. For robust stability, the form of the parameterized Lyapunov function is critical because the presence of the uncertainty within the Lyapunov function does not allow the uncertain parameters to be arbitrarily time-varying, which renders it less conservative for constant real parameter uncertainty than fixed quadratic Lyapunov functions [5].

In the present paper we extend the framework in [5] to explicitly account for the maximum rate of variation in the uncertain real parameters by considering a new class of Lyapunov functions having the form

$$V(x) = x^T[P + P_0(\Delta A(t))]x \tag{1}$$

where $\Delta A(t),\ t \geq 0$ denotes the time-varying uncertainty in the system dynamics. As in [5] this Lyapunov function is parameter-dependent. However, the novel feature of (11) is that the expression for $\dot{V}(x)$ involves the time rate of change of the uncertain parameters, i.e., $\Delta\dot{A}(t),\ t \geq 0$. Thus, it turns out that satisfying the negative definiteness of $\dot{V}(x)$ places a restriction on the maximum rate of variation on the uncertain real parameters $\Delta A(\cdot)$.

A related but different approach to the present paper is considered in [4]. Specifically, [4] considers an affine parameter-dependent Lyapunov function $V(x) = x^TP(\theta_1, \cdots, \theta_m)x$, where $P(\theta_1, \cdots, \theta_m) = P_0 + \sum_{i=1}^m \theta_i P_i$ and where P_0 corresponds to the nominal system, P_i are perturbations of P_0, and θ_i are uncertain possibly time-varying system parameters. For robust stability analysis [4] uses a linear matrix inequality (LMI) framework to solve for the variables P_i in the "convexified" affine parameter-dependent Lyapunov function to determine the range of robust stability. Alternatively, in the present paper we use (1) to develop an algebraic basis, in terms of a matrix Riccati equation, for designing robust feedback controllers for systems with slowly time-varying uncertainty. Furthermore, P in (1) in our approach does *not* correspond to the nominal system.

The contents of the paper are as follows. In Section II we establish notation and definitions. In Section III we present robust stability conditions for systems with time-varying rate-restricted real parameter uncertainty. In Section IV we generalize the results of Section III to nonlinear uncertainty. Specifically, we consider an absolute stability problem for a class of time-varying rate-restricted sector-bounded memoryless nonlinearities. The main result given in Theorem 4.1 generalizes the multivariable Popov criterion to slowly time-varying nonlinearities. In the single-input/single-output case Theorem 4.1 specializes to several well-known absolute stability criteria for time-varying rate-restricted nonlinearities from the classical literature [2], [3], [7], [8], [9]–[11]. Using the framework developed in Section III we proceed in Section V to give constructive sufficient conditions for robust stabilization for slowly time-varying real parameters via full-state feedback controllers. Finally, we close the paper in Section VI with conclusions.

II. NOTATION AND DEFINITIONS

In this section we establish notation and definitions. Let $\Re$ and $\mathcal{C}$ denote the real and complex numbers, let $(\ )^T$ and $(\ )^*$ denote transpose and complex conjugate transpose, let I_n or I denote the $n \times n$ identity matrix, and let 0_n denote the $n \times n$ zero matrix. Furthermore, $M \geq 0$ $(M > 0)$ denotes the fact that the Hermitian matrix M is nonnegative (positive) definite. An *asymptotically stable transfer function* is a transfer function each of whose poles is in the open left half-plane. The space of asymptotically stable transfer functions is denoted by $\Re\mathrm{H}_\infty$, i.e., the real-rational subset of H_∞. Let

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

denote a state-space realization of a transfer function $G(s)$, that is, $G(s) = C(sI - A)^{-1}B + D$. The notation "$\overset{\min}{\sim}$" is used to denote a minimal realization. In addition, the parahermitian conjugate $G^{\sim}(s)$ of $G(s)$ has the realization

$$G^{\sim}(s) \sim \left[\begin{array}{c|c} -A^T & C^T \\ \hline -B^T & D^T \end{array}\right].$$

Finally, the Hermitian part of G is given by $\mathrm{He}\, G = \frac{1}{2}(G + G^*)$.

A square transfer function $G(s)$ is *generalized positive real* [1] if $G(s)$ has no imaginary poles and $\mathrm{He}\, G(\jmath\omega)$ is nonnegative definite for all $\omega \in \Re$. A square transfer function $G(s)$ is *strictly generalized positive real* [1] if $G(s)$ has no imaginary poles and $\mathrm{He}\, G(\jmath\omega)$ is positive definite for all $\omega \in \Re$. A square transfer function $G(s)$ is *strongly generalized positive real* if it is strictly generalized positive real and $D + D^T > 0$, where $D \triangleq G(\infty)$. Note that a minimal realization of a generalized positive real transfer function may be unstable.

III. ROBUST STABILITY FOR SYSTEMS WITH TIME-VARYING RATE-RESTRICTED REAL PARAMETER UNCERTAINTY

In this section we address the robust stability problem for systems with time-varying rate-restricted real parameter uncertainty. Specifically, consider the nth-order uncertain system

$$\dot{x}(t) = (A + \Delta A(t))x(t), \quad t \in [0, \infty) \tag{2}$$

where $\Delta A(\cdot) \in \mathcal{U}$ and $\mathcal{U}$ is the uncertainty set defined by

$$\mathcal{U} \triangleq \{\Delta A(\cdot) : \Delta A(t) = -B_0 F(t) C_0, \quad F(\cdot) \in \mathcal{F}\} \tag{3}$$

where $\mathcal{F}$ satisfies

$$\begin{aligned} \mathcal{F} \triangleq \Bigg\{ & F : \Re^+ \to \Re^{m\times m} : \text{the elements of } F(\cdot) \text{ are Lebesgue} \\ & \text{measurable}, F(\cdot) \text{ is differentiable},\ 0 \leq F(t) \leq M, \\ & \frac{1}{2}\frac{dF(t)}{dt} \leq F(t)N_1 + F(t)N_2F(t) + N_3,\ \text{a.a. } t \geq 0 \Bigg\} \end{aligned} \tag{4}$$

and where $B_0 \in \Re^{n\times m}$, $C_0 \in \Re^{m\times n}$ are fixed matrices denoting the structure of the uncertainty, $F(t),\ t \geq 0$ is an uncertain symmetric matrix function, $M \in \Re^{m\times m}$ is a given positive-definite matrix, and N_1, N_2, and N_3 are given $m \times m$ matrices. Note that $\mathcal{F}$ may consist of block-structured matrix functions $F(t) = \mathrm{block\text{-}diag}(I_{l_1} \otimes F_1(t), I_{l_2} \otimes F_2(t), \cdots, I_{l_r} \otimes F_r(t)),\ t \geq 0$, with possibly repeated blocks so that $l_i \geq 1,\ F_i(\cdot) \in \Re^{m_i\times m_i}$, and $\sum_{i=1}^r l_i m_i = m$, where $\otimes$ denotes Kronecker product. Furthermore, we assume that $M \in \mathcal{F}$.

Remark 3.1: Note that the condition

$$\frac{1}{2}\frac{dF(t)}{dt} \leq F(t)N_1 + F(t)N_2F(t) + N_3 \tag{5}$$

in (4) provides a measure of the maximum rate of variation in the uncertain real parameters. Since N_1, N_2, and N_3 can be independently set to zero, either condition on the right-hand side of (5) can be enforced. Specifically, setting $N_1 = N_2 = 0$ (5) guarantees that the time rate of change of $F(t),\ t \geq 0$ is bounded from above by a constant matrix, i.e.,

$$\frac{1}{2}\frac{dF(t)}{dt} \leq N_3. \tag{6}$$

Alternatively, setting $N_2 = N_3 = 0$ and $N_1 = \alpha I$, where $\alpha \in \Re$, (5) guarantees that the time rate of change of $F(t), t \geq 0$ is bounded from above by $\alpha F(t),\ t \geq 0$, i.e.,

$$\frac{1}{2}\frac{dF(t)}{dt} \leq \alpha F(t). \tag{7}$$

Condition (7) can be used to enforce a functional constraint on the uncertain parameters $F(\cdot)$. For example, in aerospace applications involving propulsion systems where mass is being ejected from the system, condition (7) is clearly appropriate with $F(t),\ t \geq 0$ representing the ejected (uncertain) mass and $\alpha < 0$ representing the decay rate. Finally, note that setting $N_3 = 0,\ N_1 = \alpha I$, and $N_2 = -\alpha M^{-1}$ where $\alpha \in \Re$ yields a more restrictive condition than (6) since in this case $\frac{dF(t)}{dt} \to 0$ requires that $F(t) \to 0$ or $F(t) \to M$.

Next, we define a set $\mathcal{N}$ such that the product of the transpose of every matrix in $\mathcal{N}$ and every matrix function in $\mathcal{F}$ is uniformly nonnegative definite by

$$\mathcal{N} \triangleq \{N \in \Re^{m\times m} : N^T F(t) = F(t)N \geq 0,\ F(\cdot) \in \mathcal{F}\}. \tag{8}$$

Remark 3.2: The condition that $N^T F(t) = F(t)N,\ F(\cdot) \in \mathcal{F}$ represents an intimate relationship between the matrix N and the structure of $\mathcal{F}$. It is easy to see that there always exists a matrix N satisfying (8). For example if $F(t) = F_1(t)I_m$, where $F_1(t)$, $t \geq 0$ is a scalar time-varying uncertain parameter, then N can be an arbitrary symmetric matrix. Alternatively, if $F(t),\ t \geq 0$ is nondiagonal, then one can always choose $N = N_0 I_m$, where N_0 is a scalar. Of course, $F(\cdot)$ and N may have a more intricate structure; for example, they may be block diagonal with commuting blocks situated on the diagonal (see [5] and [6] for further details on the set $\mathcal{N}$).

For convenience we shall say that $A + \Delta A(\cdot)$ is asymptotically stable if the zero solution of the time-varying system $\dot{x}(t) = (A + \Delta A(t))x(t)$ is asymptotically stable. The next result provides a sufficient condition for the robust stability of (2) for all $\Delta A(\cdot) \in \mathcal{U}$.

Theorem 3.1: Let $N_1,\ N_2$, and N_3 be given $m \times m$ matrices, and assume $N \in \mathcal{N}$ such that $N_3^T N$ is nonnegative definite and $\hat{R} \triangleq (M^{-1} + NC_0B_0 - NN_2) + (M^{-1} + NC_0B_0 - NN_2)^T > 0$. Furthermore, assume that there exist $n \times n$ positive definite matrices P and R such that

$$\begin{aligned} 0 = {} & A^T P + PA + 2C_0^T N_3^T N C_0 \\ & + \left[B_0^T P - (I + NN_1)C_0 - NC_0A\right]^T \\ & \cdot \hat{R}^{-1}\left[B_0^T P - (I + NN_1)C_0 - NC_0A\right] + R. \end{aligned} \tag{9}$$

Then, for all $\Delta A(\cdot) \in \mathcal{U}$, the uncertain system (2) is asymptotically stable with the Lyapunov function

$$V(x) = x^T\left[P + C_0^T N^T F(t) C_0\right]x. \tag{10}$$

Proof: In order to prove the asymptotic stability of (2) for all $\Delta A(\cdot) \in \mathcal{U}$, consider the Lyapunov function candidate given by (10). Since P is positive definite and $N \in \mathcal{N}$, it follows that $V(x)$ defined by (10) is positive definite for all nonzero x. The corresponding Lyapunov derivative is given by

$$\begin{aligned} \dot{V}(x) = {} & \dot{x}^T\left[P + C_0^T N^T F C_0\right]x + x^T\left[P + C_0^T N^T F C_0\right]\dot{x} \\ & + x^T C_0^T N^T \dot{F} C_0 x \end{aligned} \tag{11}$$

or, equivalently, using (2) and the fact that $\Delta A(\cdot) \in \mathcal{U}$

$$\begin{aligned} \dot{V}(x) = x^T\big[&[A^T P + PA] - C_0^T F\left[B_0^T P - NC_0A\right] \\ & - \left[B_0^T P - NC_0A\right]^T F C_0 - C_0^T F[NC_0B_0 \\ & + (NC_0B_0)^T]FC_0 + C_0^T N^T \dot{F} C_0\big]x. \end{aligned} \tag{12}$$

Next, since $F(\cdot) \in \mathcal{F}$, (12) becomes

$$\begin{aligned} \dot{V}(x) \leq x^T\big[&[A^T P + PA] - C_0^T F\left[B_0^T P - NC_0A\right] \\ & - \left[B_0^T P - NC_0A\right]^T F C_0 - C_0^T F[NC_0B_0 \\ & + (NC_0B_0)^T]FC_0 + 2C_0^T N^T \\ & \times [FN_1 + FN_2F + N_3]C_0\big]x. \end{aligned} \tag{13}$$

Now, noting that $0 \leq F(t) \leq M,\ t \geq 0$, for all $F(\cdot) \in \mathcal{F}$ is equivalent to $F(t)MF(t) \leq F(t),\ t \geq 0$, for all $F(\cdot) \in \mathcal{F}$ [6], and adding and subtracting $x^T C_0^T[FM^{-1}F - F]C_0x$ to and from (13) and grouping terms, yields

$$\begin{aligned} \dot{V}(x) \leq x^T\big[&[A^T P + PA + 2C_0^T N_3^T N C_0] \\ & - C_0^T F\left[B_0^T P - (I + NN_1)C_0 - NC_0A\right] \\ & - \left[B_0^T P - (I + NN_1)C_0 - NC_0A\right]^T F C_0 \\ & - C_0^T F \hat{R} F C_0\big]x + x^T C_0^T[FM^{-1}F - F]C_0x \end{aligned}$$

or, equivalently

$$\dot{V}(x) \leq -x^T[R + z^T z]x + x^T C_0^T[FM^{-1}F - F]C_0x \tag{14}$$

where

$$\begin{aligned} z &\triangleq \hat{R}^{-1/2}\hat{B} + \hat{R}^{1/2}FC_0 \\ \hat{B} &\triangleq B_0^T P - (I + NN_1)C_0 - NC_0A. \end{aligned}$$

Since R is positive definite and $F(\cdot) \in \mathcal{F}$, it follows that $\dot{V}(x)$ is negative definite. Hence (2) is asymptotically stable for all $\Delta A(\cdot) \in \mathcal{U}$. □

Remark 3.3: Note that the Lyapunov function $V(x)$ given by (10) is explicitly dependent on the uncertain parameters $F(t), t \geq 0$. In the terminology of [5], this is a parameter-dependent Lyapunov function. Since the parameter-dependent Lyapunov function $V(x)$ explicitly depends on the uncertain time-varying parameters $F(t),\ t \geq 0$, it enforces an *a priori* restriction on the allowable time-variation of the uncertain parameters. Specifically, if $F(t),\ t \geq 0$ were permitted to be arbitrarily time-varying, then the terms involving $\dot{F}(t),\ t \geq 0$ would potentially subvert the negative definiteness of $\dot{V}(x)$.

Remark 3.4: Theorem 3.1 presents constructive sufficient conditions for robust stability of continuous-time systems with time-varying rate-restricted real parameter uncertainty. The conservatism of this theorem is difficult to predict. The overbounding in the proof of Theorem 3.1 needed for guaranteeing robust stability holds with respect to partial ordering of nonnegative definite matrices for which no scalar measure of conservatism is available. The conservatism will then depend upon the actual value of P determined by solving (9).

Remark 3.5: In the case where the only information available about the rate of time variation of the uncertainty is $\frac{dF(t)}{dt} \leq 0$, $t \geq 0$ asymptotic stability of (2) can be guaranteed with $N_1 = N_2 = N_3 = 0$ in Theorem 3.1. In this case (9) specializes to

$$\begin{aligned} 0 = {} & A^T P + PA + \left[B_0^T P - C_0 - NC_0A\right]^T \\ & \times \hat{R}^{-1}\left[B_0^T P - C_0 - NC_0A\right] + R. \end{aligned}$$

Remark 3.6: The existence of the Riccati equation (9) can be guaranteed by invoking a strong generalized positive real condition on

$$\begin{aligned}\mathcal{G}(s) \triangleq M^{-1} - NN_2 + (I + NN_1 + Ns)G(s) \\ - G^{\sim}(s)N_3^T NG(s)\end{aligned} \tag{15}$$

where

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B_0 \\ \hline C_0 & 0 \end{array}\right]$$

and A is asymptotically stable. Specifically, $\mathcal{G}(s)$ is strictly generalized positive real if and only if there exist matrices P, L, and W such that

$$0 = A^T P + PA + 2C_0^T N_3^T NC_0 + L^T L \tag{16}$$

$$0 = B_0^T P - (I + NN_1)C_0 - NC_0A + W^T L \tag{17}$$

$$0 = \hat{R} - W^T W \tag{18}$$

are satisfied. Now, the asymptotic stability of A and the observability of (A, C_0) guarantee that P is positive definite. Furthermore, (16)–(18) along with $\hat{R} > 0$ are equivalent to the Riccati equation (9). Hence, strong generalized positive realness of $\mathcal{G}(s)$ guarantees the existence of positive definite matrices P and R satisfying (9).

IV. ABSOLUTE STABILITY CRITERION FOR SYSTEMS WITH TIME-VARYING RATE-RESTRICTED NONLINEARITIES

In this section we provide a generalization of the results of Section III to nonlinear uncertainty. Specifically, we consider the absolute stability problem for a class Φ of time-varying rate-restricted sector-bounded nonlinearities $\phi : \Re^m \times \Re^+ \to \Re^m$. Specifically, given

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

we derive conditions that guarantee global asymptotic stability of the negative feedback interconnection of $G(s)$ and $\phi(\cdot,\cdot)$ for all $\phi(\cdot,\cdot) \in \Phi$. Note that the negative feedback interconnection of $G(s)$ and $\phi(\cdot,\cdot)$ has the state-space representation

$$\dot{x}(t) = Ax(t) - B\phi(y,t), \quad t \in [0, \infty) \tag{19}$$

$$y(t) = Cx(t). \tag{20}$$

To state the main result of this section, the following definitions are needed. Let $M \in \Re^{m\times m}$ be a given positive-definite diagonal matrix and $N_1, N_2, N_3 \in \Re^{m\times m}$ be given diagonal matrices. Next, define the set Φ of allowable nonlinearities $\phi(\cdot,\cdot)$ by (21), as shown at the bottom of the page, where $y_i = C_ix$ and C_i denotes the ith row of C.

Remark 4.1: Note that $\phi^T(y,t)[M^{-1}\phi(y,t) - y] \le 0$ and $\sum_{i=1}^m \int_0^{y_i} \frac{\partial}{\partial t}\phi_i(\sigma,t)\,d\sigma \le \phi^T(y,t)N_1y + \phi^T(y,t)N_2\phi(y,t) + y^T N_3 y$ in (21) are implied by the scalar conditions

$$0 \le \phi_i(y_i,t)y_i \le M_{ii}y_i^2, \qquad y_i \in \Re, \qquad i = 1, \cdots, m \tag{22}$$

and

$$\begin{aligned}&\int_0^{y_i} \frac{\partial}{\partial t}\phi_i(\sigma,t)\,d\sigma \\ &\quad \le \phi_i(y_i,t)N_{1_{ii}}y_i + \phi_i(y_i,t)N_{2_{ii}}\phi_i(y_i,t) + y_iN_{3_{ii}}y_i, \\ &\qquad\qquad y_i \in \Re, \qquad i = 1, \cdots, m.\end{aligned} \tag{23}$$

Condition (23) provides a measure of the maximum rate of time variation on each component of the nonlinearity $\phi(y,t)$.

For convenience in stating the next result define the notation

$$\bar{R} \triangleq (M^{-1} + NCB - NN_2) + (M^{-1} + NCB - NN_2)^T \tag{24}$$

$$\bar{B} \triangleq B^T P - (I + NN_1)C - NCA. \tag{25}$$

Theorem 4.1: Let

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

and let N_1, N_2, N_3, and N be given $m \times m$ diagonal matrices. Furthermore, assume A is asymptotically stable, N, N_3 are nonnegative definite, and $\bar{R} > 0$. Then

$$\begin{aligned}\mathcal{G}(s) \triangleq M^{-1} - NN_2 + (I + NN_1 + Ns)G(s) \\ - G^{\sim}(s)N_3N, G(s)\end{aligned} \tag{26}$$

is strongly generalized positive real if and only if there exist positive definite matrices P and R such that

$$0 = A^T P + PA + 2C^T N_3NC + \bar{B}^T\bar{R}^{-1}\bar{B} + R. \tag{27}$$

In this case

$$V(x) = x^T Px + 2\sum_{i=1}^m \int_0^{y_i} \phi_i(\sigma,t)N_{ii}\,d\sigma \tag{28}$$

where $y_i = C_ix$ is a Lyapunov function that guarantees that the negative feedback interconnection of $G(s)$ and $\phi(\cdot,\cdot)$ is asymptotically stable for all $\phi(\cdot,\cdot) \in \Phi$.

Proof: If $\mathcal{G}(s)$ is strongly generalized positive real, it follows from the spectral factorization theory that there exists a spectral factor $\mathcal{M}(s)$ such that $\mathcal{G}(s) + \mathcal{G}^{\sim}(s) = \mathcal{M}^{\sim}(s)\mathcal{M}(s)$, $s = \jmath\omega$, where $\mathcal{M}^{\pm 1}(s) \in \Re\mathrm{H}_\infty$ and $\omega \in \Re$. The existence of positive definite matrices P and R satisfying (27) now follows from algebraic state-space realization manipulations, the asymptotic stability of A, and the observability of (A, C). Conversely, using algebraic manipulations it can be shown that (27) implies that $\mathcal{G}(s)$ is strongly generalized positive real.

Next, for $\phi(\cdot,\cdot) \in \Phi$ consider the Lyapunov function candidate (28). First note that since P is positive definite and $\phi(\cdot,\cdot) \in \Phi$, $V(x)$

$$\begin{aligned}\Phi \triangleq \Bigg\{&\phi : \Re^m \times \Re^+ \to \Re^m : \phi(y,t) = [\phi_1(y_1,\cdot), \cdots, \phi_m(y_m,\cdot)],\ \phi^T(y,t)[M^{-1}\phi(y,t) - y] \le 0,\ \phi(\cdot,\cdot) \text{ is differentiable}, \\ &\sum_{i=1}^m \int_0^{y_i} \frac{\partial}{\partial t}\phi_i(\sigma,t)\,d\sigma \le \phi^T(y,t)N_1y + \phi^T(y,t)N_2\phi(y,t) + y^T N_3 y, y \in \Re^m,\ \text{a.a. } t \ge 0,\ \text{and } \phi(y,\cdot) \text{ is Lebesgue} \\ &\text{measurable for all } y \in \Re^m\Bigg\}\end{aligned} \tag{21}$$

defined by (28) is positive for all nonzero x. Thus, the corresponding Lyapunov derivative is given by

$$\begin{aligned}\dot{V}(x) &= x^T[A^TP+PA]x - \phi^TB^TPx - x^TPBx\\ &\quad + 2\sum_{i=1}^{m}\phi_i(y_i,t)N_{ii}\dot{y}_i\\ &\quad + 2\sum_{i=1}^{m}\int_0^{y_i}\frac{\partial}{\partial t}\phi_i(\sigma,t)N_{ii}\,d\sigma \qquad (29)\end{aligned}$$

or, equivalently, using (21) and $\dot{y} = CAx - CB\phi$, (29) becomes

$$\begin{aligned}\dot{V}(x) &\le x^T[A^TP+PA]x - \phi^T[B^TP - NCA]x\\ &\quad - x^T[B^TP-NCA]^T\phi - \phi^T[NCB+(NCB)^T]\phi\\ &\quad + 2\phi^TNN_1y + 2\phi^TNN_2\phi + 2y^TN_3Ny. \qquad (30)\end{aligned}$$

Adding and subtracting $2\phi^T[M^{-1}\phi - y]$ to and from (30) and grouping terms yields

$$\begin{aligned}\dot{V}(x) &\le x^T[A^TP+PA+2C^TN_3NC]x\\ &\quad - \phi^T[B^TP-(I+NN_1)C-NCA]x\\ &\quad - x^T[B^TP-(I+NN_1)C-NCA]^T\phi\\ &\quad - \phi^T\bar{R}\phi + 2\phi^T[M^{-1}\phi - y]\end{aligned}$$

or, equivalently

$$\dot{V}(x) \le -x^TRx - v^Tv + 2\phi^T[M^{-1}\phi - y] \qquad (31)$$

where $v \triangleq \bar{R}^{-1/2}\bar{B}x + \bar{R}^{1/2}\phi$. Since R is positive definite and $\phi^T(y,\cdot)[M^{-1}\phi(y,\cdot) - y] \le 0$ it follows that $\dot{V}(x)$ is negative definite. Hence global asymptotic stability of feedback interconnection of $G(s)$ and $\phi(\cdot,\cdot)$ is established for all $\phi(\cdot,\cdot) \in \Phi$. □

Remark 4.2: A similar proof using the three-equation form (16)–(18) with B_0, C_0 replaced by B, C, respectively, of the generalized positive real condition (26) can also be constructed. In this case (26) in Theorem 4.1 can be relaxed from strongly generalized positive real to strictly generalized positive real.

Remark 4.3: Theorem 4.1 is a generalization of a similar result by Rekasius and Rowland [7] and Srinath *et al.* [8] for single-input/single-output systems to multi-input/multi-output systems.

Remark 4.4: In the special case where $\phi(y,t) = F(t)y$, Theorem 4.1 specializes to Theorem 3.1 for the case of diagonal time-varying rate-restricted uncertainty $F(t)$, $t \ge 0$. However, the results of Section III allow for fully coupled uncertain rate-restricted uncertainty $F(t)$, $t \ge 0$, which cannot be addressed by means of the nonlinear theory.

Remark 4.5: Setting $N_1 = N_2 = N_3 = 0$ in (26), Theorem 4.1 specializes to the multivariable Popov criterion [5].

Remark 4.6: It is interesting to note that in the case where the only information available about the rate of the time variation of the nonlinearity is

$$\sum_{i=1}^{m}\int_0^{y_i}\frac{\partial}{\partial t}\phi_i(\sigma,t)\,d\sigma \le 0 \qquad (32)$$

asymptotic stability of (19) and (20) can be guaranteed with $N_1 = N_2 = N_3 = 0$ in Theorem 4.1. In this case $\mathcal{G}(s)$ given by (26) specializes to the familiar Popov condition

$$\mathcal{G}_{\mathrm{P}}(s) \triangleq M^{-1} + (I+Ns)G(s). \qquad (33)$$

Furthermore, in the special case where

$$\phi_i(y_i,t) = k_i(t)\hat{\phi}_i(y_i), \qquad i = 1,\cdots,m \qquad (34)$$

where $k_i(t)$, $t \ge 0$ is a linear time-varying gain and $\hat{\phi}_i(y_i)$ is a first and third quadrant time-invariant sector-bounded memoryless nonlinearity, (32) is satisfied if $\frac{dk_i(t)}{dt} \le 0$, $t \ge 0$. Hence, for this class of time-varying rate-restricted nonlinearities the Popov criterion provides sufficient conditions for absolute stability.

V. ROBUST STABILIZATION FOR SYSTEMS WITH TIME-VARYING RATE-RESTRICTED REAL PARAMETER UNCERTAINTY

In this section we consider the robust stabilization problem for systems with time-varying rate-restricted real parameter uncertainty. The problem involves the set $\mathcal{U}$ given by (3) of uncertain perturbations $\Delta A(\cdot)$ of the nominal (A,B) system. The goal of the robust stabilization problem is to determine a state feedback controller that stabilizes the plant for all variations in $\mathcal{U}$.

Robust Stabilization Problem: Determine $K \in \Re^{\hat{m}\times n}$ such that the closed-loop system consisting of the nth-order controlled plant

$$\dot{x}(t) = (A+\Delta A(t))x(t) + Bu(t), \quad t \in [0,\infty) \qquad (35)$$

and the state feedback controller

$$u(t) = Kx(t) \qquad (36)$$

is asymptotically stable for all $\Delta A(\cdot) \in \mathcal{U}$.

For each uncertain variation $\Delta A(\cdot) \in \mathcal{U}$, the closed-loop system can be written as

$$\dot{x}(t) = (A+BK+\Delta A(t))x(t), \quad t \in [0,\infty). \qquad (37)$$

The following result gives a sufficient condition for constructing a state feedback gain K that solves the robust stabilization problem with time-varying rate-restricted real parameter uncertainty. For the statement of this result let R_1 and R_2 be $n \times n$ and $\hat{m} \times \hat{m}$ positive definite matrices, respectively. Furthermore, for notational convenience recall the definition of $\hat{R}$ and define

$$\begin{aligned}P_a &\triangleq B^TP - B^TC_0^TN^T\hat{R}^{-1}\hat{B}\\ R_{2a} &\triangleq R_2 + B^TC_0^TN^T\hat{R}^{-1}NC_0B\\ \tilde{C} &\triangleq (I+NN_1)C_0 + NC_0A\\ A_P &\triangleq A - B_0\hat{R}^{-1}\tilde{C}\end{aligned}$$

for arbitrary $P \in \Re^{n\times n}$.

Theorem 5.1: Assume $\hat{R} > 0$, $N \in \mathcal{N}$, and $N_3^TN \ge 0$. Furthermore, suppose there exists an $n \times n$ positive-definite matrix P satisfying

$$\begin{aligned}0 &= A_P^TP + PA_P + R_1 + 2C_0^TN_3^TNC_0\\ &\quad + \tilde{C}^T\hat{R}^{-1}\tilde{C} + PB_0\hat{R}^{-1}B_0^TP - P_a^TR_{2a}^{-1}P_a \qquad (38)\end{aligned}$$

and let K be given by

$$K = -R_{2a}^{-1}P_a. \qquad (39)$$

Then $A+BK+\Delta A(\cdot)$ is asymptotically stable for all $\Delta A(\cdot) \in \mathcal{U}$.

Proof: With K given by (39), it follows that (38) is equivalent to

$$\begin{aligned}0 &= (A+BK)^TP + P(A+BK) + 2C_0^TN_3^TNC_0 + R_1\\ &\quad + \left[B_0^TP - (I+NN_1)C_0 - NC_0(A+BK)\right]^T\\ &\quad \times \hat{R}^{-1}\left[B_0^TP - (I+NN_1)C_0 - NC_0(A+BK)\right].\end{aligned}$$

It now follows from Theorem 3.1 that $A+BK+\Delta A(\cdot)$ is asymptotically stable for all $\Delta A(\cdot) \in \mathcal{U}$. □

Remark 5.1: Theorem 5.1 presents sufficient conditions for designing robust full-state feedback controllers for time-varying rate-restricted real parameter uncertainty. Using the fixed-structure controller synthesis framework developed in [5], these results can be readily extended to fixed-order (i.e., full- and reduced-order) dynamic compensation.

VI. CONCLUSION

In this paper we developed a new class of parameter-dependent Lyapunov functions that explicitly depend on the time variation of the uncertain parameters to guarantee robust stability in the presence of time-varying rate-restricted plant uncertainty. Extensions to a class of time-varying nonlinear uncertainty that generalizes the multivariable Popov criterion were also developed. Finally, using the parameterized Lyapunov function framework developed in the analysis part of the paper constructive sufficient conditions for robust full-state feedback control design were obtained for systems involving slowly time-varying real parameter uncertainty.

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Passivity and Disturbance Attenuation via Output Feedback for Uncertain Nonlinear Systems

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***Abstract*—The authors address the problem of disturbance attenuation with internal stability via output feedback for a class of passive systems with uncertainties. The problem is approached by means of adaptive output feedback control which does not require any state observer. The results obtained extend an earlier result of Steinberg–Corless [22]. Sufficient conditions are proposed under which a nonlinear system can be made locally or globally passive via output feedback.**

***Index Terms*— Adaptive control, disturbance attenuation, nonlinear systems, output feedback, passivity.**

I. INTRODUCTION

Passive systems theory can be traced back to the beginning of the 1970's, and its use in the feedback stabilization of nonlinear systems has recently gained renewed attention (see Byrnes *et al.* [4] and the references quoted therein). In particular, the question of when a finite-dimensional nonlinear system can be rendered passive via state feedback was solved in [4]. This important result, together with other fundamental dissipativity tools, highlights the use of passivity concepts in various interesting nonlinear control problems and applications.

The purpose of this paper is to study the nonlinear output feedback disturbance attenuation problem for passive systems in the presence of parametric uncertainties. We assume that the output used for feedback is not corrupted by disturbances. The problem to be addressed in this paper is how to find a dynamic output feedback controller which makes the system dissipative with respect to a specific supply rate while maintaining the internal stability such as Lagrange stability and asymptotic stability. Actually this is the central question in the nonlinear output-feedback H_∞-control problem. This issue has been tackled by several authors at a very general level; see recent papers [10], [2], [16], [3], and references therein. The proposed output-feedback H_∞ results are based on a "separation principle" and a corresponding state-feedback H_∞ solution [23] which requires solution of a Hamilton–Jacobi equation or inequality. The observation-based state estimation question is addressed separately. Our approach is close to the work of Battilotti [3] where the disturbance attenuation problem with stability via output feedback is solved for a class of nonlinear systems which are linear in the unmeasured states. Here, we will consider a particular class of nonlinear systems with matched nonlinear uncertainties. However, due to the passive nature of the nominal system, we remove the restriction of the vector fields depending linearly on the unmeasured states. The latter assumption was also used in a number of works in output-feedback stabilization (see, e.g., [17], [13], and [20], although [20] slightly relaxes this assumption). Our adaptive output feedback H_∞ controller is not based on any state observer. The results obtained are an extension of an earlier result of Steinberg and Corless [22] where the nominal system is a strictly positive real linear system. On the other hand, we give a solution to the problem

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