## VII. Conclusion

The parameter-dependent Lyapunov function approach of [9]–[11] for robust controller synthesis with constant real parameter uncertainty was extended to account for $H_\infty$-disturbance rejection. Specifically, by merging the results of [1] and [9]–[11], controller synthesis design equations are presented that guarantee robust stability and robust mixed $H_2/H_\infty$ performance over a specified range of constant real parameter uncertainty.

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# Robust, Reduced-Order Modeling for State-Space Systems via Parameter-Dependent Bounding Functions

Wassim M. Haddad and Vikram Kapila

***Abstract*— One of the most important problems in dynamic systems theory is to approximate a higher-order system model with a low-order, relatively simpler model. However, the nominal high-order model is never an exact representation of the true physical system. In this paper the problem of approximating an uncertain high-order system with constant real parameter uncertainty by a robust reduced-order model is considered. A *parameter-dependent* quadratic bounding function is developed that bounds the effect of uncertain real parameters on the model-reduction error. An auxiliary minimization problem is formulated that minimizes an upper bound for the model-reduction error. The principal result is a necessary condition for solving the auxiliary minimization problem which effectively provides sufficient conditions for characterizing robust reduced-order models.**

***Index Terms*— Real parameter uncertainty, reduced-order modeling, uncertain systems.**

## Nomenclature

| | |
|---|---|
| $\Re$, $\Re^{r\times s}$, $\Re^r$ | Real numbers, $r\times s$ real matrices, $\Re^{r\times 1}$. |
| $()^T$, $()^{-1}$, $\operatorname{tr}()$, $\mathcal{E}$ | Transpose, inverse, trace, expectation. |
| $I_r$, $0_r$ | $r\times r$ identity matrix, $r\times r$ zero matrix. |
| $\mathcal{S}^r$, $\mathcal{N}^r$, $\mathcal{P}^r$ | $r\times r$ symmetric, nonnegative-definite, positive-definite matrices. |
| $Z_1 \le Z_2$, $Z_1 < Z_2$ | $Z_2 - Z_1 \in \mathcal{N}^r$, $Z_2 - Z_1 \in \mathcal{P}^r$; $Z_1, Z_2 \in \mathcal{S}^r$. |
| $n, l, m, n_m, \tilde{n}$ | Positive integers; $1 \le n_m \le n$; $\tilde{n} = n + n_m$. |
| $x, y, x_m, y_m, \tilde{x}$ | $n-$, $l-$, $n_m-$, $l-$, $\tilde{n}-$ dimensional vectors. |
| $A, \Delta A; B, C$ | $n\times n; n\times m, l\times n$ matrices. |
| $A_m, B_m, C_m$ | $n_m\times n_m$, $n_m\times m$, $l\times n_m$ matrices. |
| $R$ | Model-reduction error-weighting matrix, $R\in\mathcal{P}^l$. |
| $w(\cdot)$, $V$ | $m$-dimensional white noise, intensity of $w(\cdot)$, $V\in\mathcal{P}^m$. |

## I. Introduction

One of the most important problems in dynamic systems theory is to approximate a higher-order system model with a low-order, relatively simpler model [8], [9], [11], [12]. However, the nominal high-order model is never an exact representation of the true physical system. This necessitates design tools that allow robust reduced-order modeling with respect to uncertainties in the high-order model. In many physical systems, such as flexible structures with uncertain frequency and damping, these uncertainties are characterized as highly structured, constant real parametric errors. Hence, to guarantee the best performance possible in the presence of these uncertainties it

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is important that the reduced-order model nonconservatively account for these errors.

In a recent series of papers [2]–[5], [7], a parameter-dependent bounding function framework was developed to address the problem of constant real parameter uncertainty for controller analysis and synthesis. Since the uncertain parameters appear explicitly in the parameter-dependent bounding functions, the ability of such a framework to guarantee robust stability with respect to arbitrary time-varying parameter variations is curtailed, thus reducing conservatism with respect to constant real parameter uncertainty. Furthermore, this framework naturally allows for the introduction of scaling matrices that account for structure in the uncertainty.

In this paper, we use the parameter-dependent bounding function framework developed in [2]–[5] and [7] to obtain robust reduced-order models over a specified range of constant real parametric uncertainty. The main idea is to bound the effect of the uncertain parameters on the model-reduction error over the uncertainty range and then determine a reduced-order model which minimizes the model-reduction error bound. The results presented herein provide constructive sufficient conditions which characterize robust reduced-order models with robust model-reduction error bounds. Therefore, the relative conservatism of the proposed construction is problem dependent. It is important to note that the relevance of the robust reduced-order modeling problem is to approximate the statistical behavior of a high-order uncertain system with real parametric errors; no claim whatsoever is made as to its usefulness for robust reduced-order controller design. Finally, it should be noted that these results can be used as a basis for system identification within an adaptive control framework. Specifically, since the state space (full- or reduced-order) model can be used to represent the input–output behavior of a given uncertain system with model parameters designed to minimize the error between the input–output behavior of actual system and the approximated model over a specified range of uncertainty, the resulting *identification model* can be used within an adaptive control setting as discussed in [10].

## II. Robust Model-Reduction Problem

Let $\mathcal{U} \subset \Re^{n\times n}$ denote the set of constant real uncertain perturbations $\Delta A$ of the nominal plant dynamics $A$.

*The Problem:* For fixed $n_m \le n$, determine $(A_m, B_m, C_m)$ such that for the system consisting of the $n$th-order disturbed plant

$$\dot{x}(t) = (A+\Delta A)x(t) + Bw(t), \qquad t \in [0, \infty) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

and $n_m$th-order model

$$\dot{x}_m(t) = A_m x_m(t) + B_m w(t) \tag{3}$$

$$y_m(t) = C_m x_m(t) \tag{4}$$

the model-reduction error criterion

$$J(A_m, B_m, C_m) \triangleq \sup_{\Delta A\in\mathcal{U}} \limsup_{t\to\infty} \frac{1}{t}\mathcal{E}\int_0^t \{y(s)-y_m(s)\}^T \cdot R\{y(s)-y_m(s)\}\,ds \tag{5}$$

is minimized.

For each reduced-order model $(A_m, B_m, C_m)$ and system variation $\Delta A \in \mathcal{U}$, the augmented system (1)–(4) is given by

$$\dot{\tilde{x}}(t) = (\tilde{A}+\Delta\tilde{A})\tilde{x}(t) + \tilde{B}w(t), \qquad t\in[0,\infty) \tag{6}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_m(t)\end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A & 0 \\ 0 & A_m\end{bmatrix}$$

$$\Delta\tilde{A} \triangleq \begin{bmatrix} \Delta A & 0 \\ 0 & 0\end{bmatrix}, \quad \tilde{B} \triangleq \begin{bmatrix} B \\ B_m\end{bmatrix}.$$

## III. Sufficient Conditions for Robust Performance

The following result is immediate.

*Lemma 3.1:* Suppose $A_m$ is asymptotically stable and $A+\Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Then

$$J(A_m, B_m, C_m) = \sup_{\Delta A\in\mathcal{U}} \operatorname{tr} \tilde{Q}_{\Delta\tilde{A}}\tilde{R} \tag{7}$$

where $\tilde{R} \triangleq [C \quad -C_m]^T R [C \quad -C_m]$ and $\tilde{Q}_{\Delta\tilde{A}} \triangleq \lim_{t\to\infty}\mathcal{E}[\tilde{x}(t)\tilde{x}^T(t)] \in \mathcal{N}^{\tilde{n}}$ is the unique, nonnegative-definite solution to

$$0 = (\tilde{A}+\Delta\tilde{A})\tilde{Q}_{\Delta\tilde{A}} + \tilde{Q}_{\Delta\tilde{A}}(\tilde{A}+\Delta\tilde{A})^T + \tilde{V} \tag{8}$$

where $\tilde{V} \triangleq \tilde{B}V\tilde{B}^T$.

We now determine an upper bound for $J(A_m, B_m, C_m)$ given by (7). The key step in obtaining robust performance is to bound the uncertain terms $\Delta\tilde{A}\tilde{Q}_{\Delta\tilde{A}} + \tilde{Q}_{\Delta\tilde{A}}\Delta\tilde{A}^T$ in the Lyapunov equation (8) by means of a *parameter-dependent* bounding function $\Omega(\tilde{Q}, \Delta\tilde{A})$. As discussed in [2]–[5] and [7], a key feature of this approach is the fact that it constrains the class of allowable time-varying uncertainties, thus reducing conservatism in the presence of constant real parameter uncertainty, and hence providing sharper $H_2$-performance bounds. The following result is fundamental and forms the basis for all later developments.

*Theorem 3.1:* Let $\Omega_0: \mathcal{N}^{\tilde{n}} \to \mathcal{S}^{\tilde{n}}$ and $\mathcal{Q}_0: \mathcal{U} \to \mathcal{S}^{\tilde{n}}$ be such that

$$\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^T \le \Omega_0(\mathcal{Q}) - [(\tilde{A}+\Delta\tilde{A})\mathcal{Q}_0(\Delta\tilde{A}) + \mathcal{Q}_0(\Delta\tilde{A})(\tilde{A}+\Delta\tilde{A})^T], \quad \Delta A\in\mathcal{U}, \quad \mathcal{Q}\in\mathcal{N}^{\tilde{n}} \tag{9}$$

and, for a given $(A_m, B_m, C_m)$, suppose there exists $\mathcal{Q} \in \mathcal{N}^{\tilde{n}}$ satisfying

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \Omega_0(\mathcal{Q}) + \tilde{V} \tag{10}$$

such that $\mathcal{Q} + \mathcal{Q}_0(\Delta\tilde{A})$ is nonnegative definite for all $\Delta A \in \mathcal{U}$. Then

$$(\tilde{A}+\Delta\tilde{A}, \tilde{V}^{1/2}) \text{ is stabilizable}, \quad \Delta A\in\mathcal{U} \tag{11}$$

if and only if

$$A_m \text{ and } A+\Delta A \text{ are asymptotically stable}, \quad \Delta A\in\mathcal{U}. \tag{12}$$

In this case

$$\tilde{Q}_{\Delta\tilde{A}} \le \mathcal{Q} + \mathcal{Q}_0(\Delta\tilde{A}), \qquad \Delta A\in\mathcal{U} \tag{13}$$

where $\tilde{Q}_{\Delta\tilde{A}}$ is given by (8). Furthermore

$$J(A_m, B_m, C_m) \le \operatorname{tr}\mathcal{Q}\tilde{R} + \sup_{\Delta A\in\mathcal{U}} \operatorname{tr}\mathcal{Q}_0(\Delta\tilde{A})\tilde{R}. \tag{14}$$

If, in addition, there exists $\overline{\mathcal{Q}}_0 \in \mathcal{S}^{\tilde{n}}$ such that

$$\mathcal{Q}_0(\Delta\tilde{A}) \le \overline{\mathcal{Q}}_0, \qquad \Delta A\in\mathcal{U} \tag{15}$$

then

$$J(A_m, B_m, C_m) \le \operatorname{tr}[(\mathcal{Q}+\overline{\mathcal{Q}}_0)\tilde{R}]. \tag{16}$$

*Proof:* We stress that in (9) $\mathcal{Q}$ denotes an arbitrary element of $\mathcal{N}^{\tilde{n}}$, whereas in (10) $\mathcal{Q}$ denotes a specific solution of the modified Lyapunov equation (10). This minor abuse of notation considerably simplifies the presentation. Now, note that for all $\Delta A \in \Re^{n\times n}$, (10) is equivalent to

$$0 = (\tilde{A}+\Delta\tilde{A})\mathcal{Q} + \mathcal{Q}(\tilde{A}+\Delta\tilde{A})^T + \Omega_0(\mathcal{Q}) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^T) + \tilde{V}. \tag{17}$$

Adding and subtracting $(\tilde{A}+\Delta\tilde{A})\mathcal{Q}_0(\Delta\tilde{A}) + \mathcal{Q}_0(\Delta\tilde{A})(\tilde{A}+\Delta\tilde{A})^T$ to and from (17) yields

$$0 = (\tilde{A}+\Delta\tilde{A})[\mathcal{Q}+\mathcal{Q}_0(\Delta\tilde{A})] + [\mathcal{Q}+\mathcal{Q}_0(\Delta\tilde{A})] \cdot (\tilde{A}+\Delta\tilde{A})^T + \Omega_0(\mathcal{Q}) - [(\tilde{A}+\Delta\tilde{A})\mathcal{Q}_0(\Delta\tilde{A}) + \mathcal{Q}_0(\Delta\tilde{A})(\tilde{A}+\Delta\tilde{A})^T] - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^T) + \tilde{V}. \tag{18}$$

Hence, by assumption, (18) has a solution $\mathcal{Q} \in \mathcal{N}^{\tilde{n}}$ for all $\Delta A \in \Re^{n\times n}$. If $\Delta A$ is restricted to the set $\mathcal{U}$, then, by (9), the expression

$$\Omega_0(\mathcal{Q}) - [(\tilde{A}+\Delta\tilde{A})\mathcal{Q}_0(\Delta\tilde{A}) + \mathcal{Q}_0(\Delta\tilde{A})(\tilde{A}+\Delta\tilde{A})^T] - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^T)$$

is nonnegative definite. Thus if the stabilizability condition holds for all $\Delta A \in \mathcal{U}$, then it follows from [13, Th. 3.6] that $\{\tilde{A}+\Delta\tilde{A}, [\tilde{V} + \Omega(\mathcal{Q}, \Delta\tilde{A}) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^T)]^{1/2}\}$ is stabilizable for all $\Delta A \in \mathcal{U}$, where

$$\Omega(\mathcal{Q}, \Delta\tilde{A}) \triangleq \Omega_0(\mathcal{Q}) - [(\tilde{A}+\Delta\tilde{A})\mathcal{Q}_0(\Delta\tilde{A}) + \mathcal{Q}_0(\Delta\tilde{A})(\tilde{A}+\Delta\tilde{A})^T]. \tag{19}$$

It now follows from (18) and [13, Lemma 12.2] that $\tilde{A}+\Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Furthermore, since $\tilde{A}+\Delta\tilde{A}$ is block diagonal (12) holds. Conversely, if (12) holds, then (11) is immediate. Now subtracting (8) from (18) yields

$$\begin{aligned} 0 =& (\tilde{A}+\Delta\tilde{A})[\mathcal{Q} + \mathcal{Q}_0(\Delta\tilde{A}) - \tilde{Q}_{\Delta\tilde{A}}] \\ &+ [\mathcal{Q} + \mathcal{Q}_0(\Delta\tilde{A}) - \tilde{Q}_{\Delta\tilde{A}}](\tilde{A}+\Delta\tilde{A})^T + \Omega_0(\mathcal{Q}) \\ &- [(\tilde{A}+\Delta\tilde{A})\mathcal{Q}_0(\Delta\tilde{A}) + \mathcal{Q}_0(\Delta\tilde{A})(\tilde{A}+\Delta\tilde{A})^T] \\ &- (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^T), \qquad \Delta A \in \mathcal{U} \end{aligned} \tag{20}$$

or, equivalently, since $\tilde{A}+\Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$

$$\begin{aligned} \mathcal{Q} + \mathcal{Q}_0(\Delta\tilde{A}) - \tilde{Q}_{\Delta\tilde{A}} =& \int_0^\infty e^{(\tilde{A}+\Delta\tilde{A})t}[\Omega(\mathcal{Q}, \Delta\tilde{A}) \\ &- (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^T)]\, e^{(\tilde{A}+\Delta\tilde{A})^T t}\, dt \\ &\geq 0, \qquad \Delta A \in \mathcal{U} \end{aligned} \tag{21}$$

which implies (13). The performance bounds (14), (16) are now an immediate consequence of (7), (13), and (15). □

Note that with $\Omega(\mathcal{Q}, \Delta\tilde{A})$ defined by (19), condition (9) can be written as

$$\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^T \leq \Omega(\mathcal{Q}, \Delta\tilde{A}), \qquad \Delta A \in \mathcal{U}, \qquad \mathcal{Q} \in \mathcal{N}^{\tilde{n}} \tag{22}$$

where $\Omega(\mathcal{Q}, \Delta\tilde{A})$ is a function of the uncertain parameters $\Delta\tilde{A}$. For convenience we shall say that $\Omega(\cdot, \cdot)$ is a parameter-dependent bounding function.

*Remark 3.1:* Theorem 3.1 provides sufficient conditions for robust reduced-order modeling with an upper bound on the modeling error. However, it is important to emphasize that our results are limited to systems which remain asymptotically stable over the class of uncertainties. Relevant applications include, for example, damped flexible structures with uncertain modal data.

## IV. Uncertainty Structure and a Parameter-Dependent Bounding Function

In this section, we assign explicit structure to the uncertainty set $\mathcal{U}$ and the parameter-dependent bounding function $\Omega(\cdot, \cdot)$. Specifically, the uncertainty set $\mathcal{U}$ is defined by

$$\mathcal{U} \triangleq \{\Delta A \in \Re^{n\times n} : \Delta A = B_0 F C_0,\ F \in \mathcal{F}\} \tag{23}$$

where $\mathcal{F}$ satisfies

$$\mathcal{F} \subseteq \hat{\mathcal{F}} \triangleq \{F \in \mathcal{S}^{m_0} : M_1 \leq F \leq M_2\} \tag{24}$$

and $B_0 \in \Re^{n\times m_0}$, $C_0 \in \Re^{m_0\times n}$ are fixed matrices denoting the structure of uncertainty, $F \in \mathcal{S}^{m_0}$ is an uncertain symmetric matrix, and $M_1, M_2 \in \mathcal{S}^{m_0}$ are symmetric matrices such that $M \triangleq M_2 - M_1$ is positive definite. Note that $M_1, M_2 \in \hat{\mathcal{F}}$. Furthermore, for flexibility, $\mathcal{F}$ may be a specified proper subset of $\hat{\mathcal{F}}$. For example, $\mathcal{F} \subseteq \hat{\mathcal{F}}$ may consist of block-structured matrices $F = \text{block-diag}(I_{l_1}\otimes F_1, I_{l_2}\otimes F_2, \cdots, I_{l_r}\otimes F_r)$ with possibly repeated blocks so that $l_i \geq 1$, $F_i \in \Re^{m_{0_i}\times m_{0_i}}$, and $\sum_{i=1}^r l_i m_{0_i} = m_0$ and where $\otimes$ denotes Kronecker product. Furthermore, we assume that $M_1, M_2 \in \mathcal{F}$. We restrict our attention to symmetric uncertainties for convenience only. More general uncertainty sets as in [2] can also be considered.

The following lemma provides an equivalent characterization of the uncertainty $F \in \mathcal{F}$.

*Lemma 4.1 [6]:* Let $F, M_1, M_2 \in \mathcal{S}^{m_0}$, and $M \in \mathcal{P}^{m_0}$. Then $F \in \mathcal{F}$ if and only if

$$(F - M_1) - (F - M_1)M^{-1}(F - M_1) \geq 0. \tag{25}$$

Using the uncertainty structure defined in (23), it follows that the augmented system (6) has structured uncertainty of the form $\Delta\tilde{A} = \tilde{B}_0 F\tilde{C}_0$, where $\tilde{B}_0 \triangleq \begin{bmatrix} B_0 \\ 0 \end{bmatrix}$ and $\tilde{C}_0 \triangleq [C_0\ \ 0]$.

Next, define the set of compatible scaling matrices $\mathcal{H}$, $\mathcal{N}_s$, and $\mathcal{N}_{\text{nd}}$ by

$$\mathcal{H} \triangleq \{H \in \mathcal{P}^{m_0} : FH = HF \text{ and } F \in \mathcal{F}\} \tag{26}$$

$$\mathcal{N}_{\text{s}} \triangleq \{N \in \Re^{m_0\times m_0} : FN^T = NF,\ F \in \mathcal{F}\} \tag{27}$$

$$\mathcal{N}_{\text{nd}} \triangleq \{N \in \Re^{m_0\times m_0} : (F - M_1)N^T = N(F - M_1) \geq 0,\ F \in \mathcal{F}\}. \tag{28}$$

We now specify the parameter-dependent bounding function $\Omega(\cdot, \cdot)$ satisfying (9).

*Proposition 4.1:* Let $H \in \mathcal{H}$, $N \in \mathcal{N}_{\text{s}}$, $M$ be positive definite, and

$$[HM^{-1} - \tilde{C}_0\tilde{B}_0 N] + [HM^{-1} - \tilde{C}_0\tilde{B}_0 N]^T > 0. \tag{29}$$

Furthermore, let $\mathcal{U}$ be defined by (23) and define $\Omega_0(\mathcal{Q})$ and $\mathcal{Q}_0(F)$ by

$$\begin{aligned} \Omega_0(\mathcal{Q}) \triangleq& [\tilde{B}_0 H + \{\tilde{A} + \tilde{B}_0 M_1\tilde{C}_0\}\tilde{B}_0 N + \mathcal{Q}\tilde{C}_0^T] \\ &\cdot [\{HM^{-1} - \tilde{C}_0\tilde{B}_0 N\} \\ &+ \{HM^{-1} - \tilde{C}_0\tilde{B}_0 N\}^T]^{-1} \\ &\cdot [\tilde{B}_0 H + \{\tilde{A} + \tilde{B}_0 M_1\tilde{C}_0\}\tilde{B}_0 N + \mathcal{Q}\tilde{C}_0^T]^T \\ &+ \tilde{B}_0 M_1\tilde{C}_0\mathcal{Q} + \mathcal{Q}\tilde{C}_0^T M_1\tilde{B}_0^T \end{aligned} \tag{30}$$

and

$$\mathcal{Q}_0(F) \triangleq \tilde{B}_0 N(F - M_1)\tilde{B}_0^T. \tag{31}$$

Then (9) is satisfied.

*Proof:* Note that by (29) $[HM^{-1} - \tilde{C}_0\tilde{B}_0 N] + [HM^{-1} - \tilde{C}_0\tilde{B}_0 N]^T > 0$. Next, since $H \in \mathcal{H}$ and $M_1, M_2 \in \mathcal{F}$, it follows that $(F - M_1)H = H(F - M_1)$ and $M^{-1}H = HM^{-1}$. Now noting that $H$ (25) = (25) $H$, it follows that $H[(F - M_1) - (F - M_1)M^{-1}(F - M_1)] \geq 0$ and thus yielding $\tilde{B}_0 H[(F - M_1) - (F -$

$M_1)M^{-1}(F-M_1)]\tilde{B}_0^T \geq 0$. Hence, it follows that

$$\begin{aligned}0 \leq\, &[\{\tilde{B}_0 H + (\tilde{A}+\tilde{B}_0 M_1 \tilde{C}_0)\tilde{B}_0 N + \mathcal{Q}\tilde{C}_0^T\} \\ &- \tilde{B}_0(F-M_1)[\{HM^{-1}-\tilde{C}_0\tilde{B}_0 N\} \\ &+ \{HM^{-1}-\tilde{C}_0\tilde{B}_0 N\}^T]] \\ &\cdot [\{HM^{-1}-\tilde{C}_0\tilde{B}_0 N\} + \{HM^{-1}-\tilde{C}_0\tilde{B}_0 N\}^T]^{-1} \\ &\cdot [\{\tilde{B}_0 H + (\tilde{A}+\tilde{B}_0 M_1 \tilde{C}_0)\tilde{B}_0 N + \mathcal{Q}\tilde{C}_0^T\} \\ &- \tilde{B}_0(F-M_1)[\{HM^{-1}-\tilde{C}_0\tilde{B}_0 N\} \\ &+ \{HM^{-1}-\tilde{C}_0\tilde{B}_0 N\}^T]]^T \\ &+ 2\tilde{B}_0 H[(F-M_1)-(F-M_1)M^{-1}(F-M_1)]\tilde{B}_0^T \\ =\, &\{\tilde{B}_0 H + (\tilde{A}+\tilde{B}_0 M_1 \tilde{C}_0)\tilde{B}_0 N + \mathcal{Q}\tilde{C}_0^T\} \\ &\cdot [\{HM^{-1}-\tilde{C}_0\tilde{B}_0 N\} + \{HM^{-1}-\tilde{C}_0\tilde{B}_0 N\}^T]^{-1} \\ &\cdot \{\tilde{B}_0 H + (\tilde{A}+\tilde{B}_0 M_1 \tilde{C}_0)\tilde{B}_0 N + \mathcal{Q}\tilde{C}_0^T\}^T - \tilde{B}_0 \\ &\cdot (F-M_1)\{\tilde{B}_0 H + (\tilde{A}+\tilde{B}_0 M_1 \tilde{C}_0)\tilde{B}_0 N + \mathcal{Q}\tilde{C}_0^T\}^T \\ &- \{\tilde{B}_0 H + (\tilde{A}+\tilde{B}_0 M_1 \tilde{C}_0)\tilde{B}_0 N + \mathcal{Q}\tilde{C}_0^T\} \\ &\cdot (F-M_1)\tilde{B}_0^T + \tilde{B}_0(F-M_1)[\{HM^{-1}-\tilde{C}_0\tilde{B}_0 N\} \\ &+ \{HM^{-1}-\tilde{C}_0\tilde{B}_0 N\}^T](F-M_1)\tilde{B}_0^T \\ &+ 2\tilde{B}_0 H[(F-M_1)-(F-M_1)M^{-1}(F-M_1)]\tilde{B}_0^T \\ =\, &\Omega_0(\mathcal{Q}) - \tilde{A}\tilde{B}_0 N(F-M_1)\tilde{B}_0^T - \tilde{B}_0(F-M_1) \\ &\cdot N^T\tilde{B}_0^T\tilde{A}^T - \tilde{B}_0 F\tilde{C}_0\tilde{B}_0 N(F-M_1)\tilde{B}_0^T \\ &- \tilde{B}_0(F-M_1)N^T\tilde{B}_0^T\tilde{C}_0^T F\tilde{B}_0^T \\ &- \tilde{B}_0 F\tilde{C}_0\mathcal{Q} - \mathcal{Q}\tilde{C}_0^T F\tilde{B}_0^T \\ =\, &\Omega_0(\mathcal{Q}) - \tilde{A}\mathcal{Q}_0(F) - \mathcal{Q}_0(F)\tilde{A}^T - \Delta\tilde{A}\mathcal{Q}_0(F) \\ &- \mathcal{Q}_0(F)\Delta\tilde{A}^T - \Delta\tilde{A}\mathcal{Q} - \mathcal{Q}\Delta\tilde{A}^T \\ =\, &\Omega_0(\mathcal{Q}) - [(\tilde{A}+\Delta\tilde{A})\mathcal{Q}_0(F) + \mathcal{Q}_0(F)(\tilde{A}+\Delta\tilde{A})^T] \\ &- \Delta\tilde{A}\mathcal{Q} - \mathcal{Q}\Delta\tilde{A}^T\end{aligned}$$

which proves (9) with $\mathcal{U}$ given by (23). □

Note that with $N \in \mathcal{N}_{\rm s}$, it follows from (24) that there exists a matrix $\mu \in \mathcal{N}^{m_0}$ such that $N(F-M_1) \leq \mu$ for all $F \in \mathcal{F}$. Next, using Theorem 3.1 and Proposition 4.1, we have the following immediate result.

*Theorem 4.1:* Let $H \in \mathcal{H}$, $N \in \mathcal{N}_{\rm nd}$, and let $M_1, M_2 \in \mathcal{S}^{m_0}$ be such that $M$ is positive definite and (29) is satisfied. Furthermore, suppose there exists a nonnegative-definite matrix $\mathcal{Q}$ satisfying

$$\begin{aligned}0 =\, &(\tilde{A}+\tilde{B}_0 M_1 \tilde{C}_0)\mathcal{Q} + \mathcal{Q}(\tilde{A}+\tilde{B}_0 M_1 \tilde{C}_0)^T \\ &+ [\tilde{B}_0 H + \{\tilde{A}+\tilde{B}_0 M_1 \tilde{C}_0\}\tilde{B}_0 N + \mathcal{Q}\tilde{C}_0^T] \\ &\cdot [\{HM^{-1}-\tilde{C}_0\tilde{B}_0 N\} + \{HM^{-1}-\tilde{C}_0\tilde{B}_0 N\}^T]^{-1} \\ &\cdot [\tilde{B}_0 H + \{\tilde{A}+\tilde{B}_0 M_1 \tilde{C}_0\}\tilde{B}_0 N + \mathcal{Q}\tilde{C}_0^T]^T + \tilde{V}. \qquad (32)\end{aligned}$$

Then $(\tilde{A}+\Delta\tilde{A}, \tilde{V}^{1/2})$ is stabilizable for all $\Delta A \in \mathcal{U}$ if and only if $A_m$ and $A+\Delta A$ are asymptotically stable for all $\Delta A \in \mathcal{U}$. In this case

$$\begin{aligned}J(A_m, B_m, C_m) \leq\, &\mathcal{J}(\mathcal{Q}, A_m, B_m, C_m) \\ &\triangleq \operatorname{tr}[(\mathcal{Q}+\tilde{B}_0\mu\tilde{B}_0^T)\tilde{R}]. \qquad (33)\end{aligned}$$

*Proof:* The result is a direct specialization of Theorem 3.1 using Proposition 4.1. We only note that $\mathcal{Q}_0(\Delta\tilde{A})$ now has the form $\mathcal{Q}_0(F) = \tilde{B}_0 N(F-M_1)\tilde{B}_0^T$. Since by assumption $N \in \mathcal{N}_{\rm nd}$ and $(F-M_1) \in \mathcal{F}$ for all $F \in \mathcal{F}$, it follows that $\mathcal{Q}+\mathcal{Q}_0(F)$ is nonnegative definite for all $F \in \mathcal{F}$ as required by Theorem 3.1. Finally, (33) is a restatement of (16) by noting that $\mathcal{Q}_0(F) = \tilde{B}_0 N(F-M_1)\tilde{B}_0^T \leq \tilde{B}_0\mu\tilde{B}_0^T$ for all $F \in \mathcal{F}$. □

*Remark 4.1:* The condition $FH = HF$, $F \in \mathcal{F}$, is analogous to the commuting assumption between the $D$-scales and $\Delta$ blocks in $\mu$-analysis which accounts for structure in the uncertainty $\Delta$. However, the condition $FN^T = NF$, for $F \in \mathcal{F}$ and $N \in \mathcal{N}_s$, has no counterpart in standard $\mu$-analysis and, as noted in [2], allows for a generalization of mixed-$\mu$ analysis to address fully populated *real* uncertain matrix blocks. Note that there always exist matrices $H \in \mathcal{H}$ and $N \in \mathcal{N}_{\rm s}$ even if $F \in \mathcal{F}$ is not diagonal. For example, if $F$ is nondiagonal, then one can choose $H = H_0 I_{m_0}$ and $N = N_0 I_{m_0}$, where $H_0$, $N_0$ are scalars. Alternatively, $F$ and $H$ and $F$ and $N$ may be block diagonal with commuting blocks situated on the diagonal. Finally, if $F = F_1 I_{m_0}$, where $F_1$ is a scalar uncertainty, then $H$ can be an arbitrary positive definite matrix and $N$ can be an arbitrary symmetric matrix. For details on specific structures of $H$ and $N$, see [6].

*Remark 4.2:* In applications it is useful to exploit the fact that $F$ may represent a fully populated uncertain matrix. To see how such multivariable uncertainty may be useful in practice, consider the multiple degree of freedom vibrational system

$$M_0\ddot{x}(t) + C_0\dot{x}(t) + (K_0+\Delta K)x(t) = 0$$

where $M_0$, $C_0$, and $K_0$ denote generalized mass, damping, and stiffness coefficients, respectively, and where $\Delta K$ denotes stiffness uncertainty. In state-space form this system can be written as

$$\dot{z}(t) = \begin{bmatrix} 0 & I \\ -M_0^{-1}(K_0+\Delta K) & -M_0^{-1}C_0 \end{bmatrix} z(t)$$

where $z(t) = [x^T(t) \quad \dot{x}(t)^T]^T$. In accordance with (23), a representation of the uncertain component of the system dynamics is thus given by

$$B_0 F C_0 = \begin{bmatrix} 0 \\ -M_0^{-1} \end{bmatrix} \Delta K [I \quad 0].$$

In this case $F = \Delta K$ in (24).

Next, we formulate the auxiliary minimization problem to minimize the error bound (33).

*Auxiliary Minimization Problem:* Determine $(\mathcal{Q}, A_m, B_m, C_m)$ with $\mathcal{Q} \in \mathcal{N}^{\tilde{n}}$, which minimizes $\mathcal{J}(\mathcal{Q}, A_m, B_m, C_m)$ subject to (32).

The relationship between the auxiliary minimization problem and the robust model-reduction problem is straightforward, as shown by the following observation.

*Proposition 4.2:* If $(\mathcal{Q}, A_m, B_m, C_m)$ satisfies (32) with $\mathcal{Q} \in \mathcal{N}^{\tilde{n}}$ and $(\tilde{A}+\Delta\tilde{A}, \tilde{V}^{1/2})$ is stabilizable, then $A_m$ and $A+\Delta A$ are asymptotically stable for $\Delta A \in \mathcal{U}$ and

$$J(A_m, B_m, C_m) \leq \mathcal{J}(\mathcal{Q}, A_m, B_m, C_m). \qquad (34)$$

*Proof:* Since (32) has a solution $\mathcal{Q} \in \mathcal{N}^{\tilde{n}}$ and $(\tilde{A}+\Delta\tilde{A}, \tilde{V}^{1/2})$ is stabilizable, the hypotheses of Theorem 3.1 are satisfied so that the robust performance bound (16) is guaranteed; (34) is merely a restatement of (16). □

## V. Sufficient Conditions for Robust Reduced-Order Modeling

To state the main result of this section involving robust, reduced-order models, we require some additional notation. Specifically, for arbitrary $Q \in \Re^{n\times n}$ define the notation

$$\begin{aligned}\tilde{V}_0 &\triangleq \{HM^{-1}-C_0B_0N\} + \{HM^{-1}-C_0B_0N\}^T \\ \mathcal{A}_Q &\triangleq (A+B_0M_1C_0) \\ &\quad + [B_0H + (A+B_0M_1C_0)B_0N]\tilde{V}_0^{-1}C_0 \\ \mathcal{A}_{\hat{Q}} &\triangleq \mathcal{A}_Q + QC_0^T\tilde{V}_0^{-1}C_0.\end{aligned}$$

*Theorem 5.1:* Let $n_m \leq n$, $H \in \mathcal{H}$, and $N \in \mathcal{N}_{\rm nd}$. Furthermore, suppose there exist $n \times n$ nonnegative-definite matrices $Q$, $\hat{Q}$, and $\hat{P}$ satisfying

$$0 = \mathcal{A}_Q Q + Q\mathcal{A}_Q^T + [B_0 H + (A + B_0 M_1 C_0)B_0 N] \cdot \tilde{V}_0^{-1}[B_0 H + (A + B_0 M_1 C_0)B_0 N]^T + QC_0^T \tilde{V}_0^{-1} C_0 Q + \tau_\perp BVB^T \tau_\perp^T \tag{35}$$

$$0 = \mathcal{A}_{\hat{Q}} \hat{Q} + \hat{Q}\mathcal{A}_{\hat{Q}}^T + \hat{Q}C_0^T \tilde{V}_0^{-1} C_0 \hat{Q} + BVB^T - \tau_\perp BVB^T \tau_\perp^T \tag{36}$$

$$0 = \mathcal{A}_{\hat{Q}}^T \hat{P} + \hat{P}\mathcal{A}_{\hat{Q}} + C^T RC - \tau_\perp^T C^T RC\tau_\perp \tag{37}$$

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_m \tag{38}$$

$$\hat{Q}\hat{P} = G^T \hat{M}\Gamma, \quad \Gamma G^T = I_{n_m}, \quad \hat{M} \in \Re^{n_m \times n_m} \tag{39}$$

$$\tau \triangleq G^T \Gamma, \quad \tau_\perp \triangleq I_n - \tau \tag{40}$$

and let $A_m$, $B_m$, $C_m$ be given by

$$A_m = \Gamma \mathcal{A}_{\hat{Q}} G^T, \quad B_m = \Gamma B, \quad C_m = CG^T. \tag{41}$$

Then $(\tilde{A} + \Delta\tilde{A}, \tilde{V}^{1/2})$ is stabilizable for all $\Delta A \in \mathcal{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. In this case the model-reduction error satisfies the bound

$$J(A_m, B_m, C_m) \leq \operatorname{tr}[(Q + B_0 \mu B_0^T)C^T RC]. \tag{42}$$

*Proof:* The proof is constructive in nature. Specifically, first we obtain necessary conditions for the auxiliary minimization problem that guarantee the existence of $n \times n$ nonnegative-definite matrices $Q$, $\hat{Q}$, and $\hat{P}$ satisfying (35)–(38). Conversely, it can be shown by construction that if there exist $n \times n$ nonnegative-definite matrices $Q$, $\hat{Q}$, and $\hat{P}$ satisfying (35)–(38), then $A_m$, $B_m$, $C_m$ given by (41) and

$$\mathcal{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^T \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^T \end{bmatrix} = \begin{bmatrix} Q & 0_{n \times n_m} \\ 0_{n_m \times n} & 0_{n_m} \end{bmatrix} + \begin{bmatrix} I_n \\ \Gamma \end{bmatrix} \hat{Q}[I_n \quad \Gamma^T] \geq 0$$

satisfy (32) with $\mathcal{J}(\mathcal{Q}, A_m, B_m, C_m)$ given by (42). Hence, it follows from Proposition 4.2 that (35)–(37) serve as sufficient conditions for robust reduced-order modeling and provide a worst case $H_2$-performance bound. □

Theorem 5.1 provides constructive sufficient conditions for the robust reduced-order modeling problem which explicitly characterize extremals $(A_m, B_m, C_m)$. These sufficient conditions consist of a system of two modified Lyapunov equations and one modified Riccati equation coupled by an oblique projection $\tau$ and uncertainty terms. Setting $B_0$ and $C_0$ to zero, i.e., deleting the parametric plant uncertainty, it can be seen that (35) drops out while (36) and (37) reduce to the optimal projection equations for model reduction obtained in [8].

*Remark 5.1:* The conservatism of bound (42) is difficult to predict for two reasons. First, the overbounding (9) holds with respect to partial ordering of nonnegative-definite matrices for which no scalar measure of conservatism is available. And, second, (9) is required to hold for all nonnegative-definite matrices $\mathcal{Q}$. The conservatism will thus depend upon the actual value of $\mathcal{Q}$ determined by solving (32). Numerical experience, however, shows that since the overbounding is parameter-dependent, it provides less conservative performance bounds as opposed to parameter-independent (i.e., $\mathcal{Q}_0(\Delta A) = 0$) bounding frameworks for constant real parameter uncertainty [5].

*Remark 5.2:* In the full order case $n_m = n$, $\tau = G = \Gamma = I_n$ [8]. In this case the robust full-order model characterized by $(\mathcal{A}_{\hat{Q}}, B, C)$ where $Q$ satisfies

$$0 = \mathcal{A}_Q Q + Q\mathcal{A}_Q^T + [B_0 H + (A + B_0 M_1 C_0)B_0 N] \cdot \tilde{V}_0^{-1}[B_0 H + (A + B_0 M_1 C_0)B_0 N]^T + QC_0^T \tilde{V}_0^{-1} C_0 Q \tag{43}$$

provides a full-order identification model that approximates the behavior of the uncertain model (1), (2) in a least-squares sense over a specified range of constant real uncertain parameters.

*Remark 5.3:* When solving (35)–(37) numerically, the matrices $M_1$, $M_2$, $H$, and $N$ and the structure matrices $B_0$ and $C_0$ appearing in the design equations can be adjusted to examine the tradeoffs between $H_2$ performance and robustness. As discussed in [5] and [7], to further reduce conservatism, one can view the scaling matrices $H$ and $N$ as free parameters and optimize the $H_2$-performance bound $\mathcal{J}(\cdot)$ with respect to $H$ and $N$. In particular, setting $\partial\mathcal{J}/\partial H = 0$ and $\partial\mathcal{J}/\partial N = 0$ yields

$$\begin{aligned} 0 &= \frac{1}{2}\frac{\partial\mathcal{J}}{\partial H} \\ &= [\tilde{B}_0 - \{(\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0)\tilde{B}_0 N + \mathcal{Q}\tilde{C}_0^T + \tilde{B}_0 H\} \\ &\quad \cdot \tilde{V}_0^{-1} M^{-1}]^T \tilde{P}[(\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0)\tilde{B}_0 N \\ &\quad + \mathcal{Q}\tilde{C}_0^T + \tilde{B}_0 H]\tilde{V}_0^{-1} \end{aligned} \tag{44}$$

$$\begin{aligned} 0 &= \frac{1}{2}\frac{\partial\mathcal{J}}{\partial N} \\ &= \tilde{B}_0[(\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0) + \{(\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0)\tilde{B}_0 N \\ &\quad + \mathcal{Q}\tilde{C}_0^T + \tilde{B}_0 H\}\tilde{V}_0^{-1}\tilde{C}_0]^T \tilde{P} \\ &\quad \cdot [(\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0)\tilde{B}_0 N + \mathcal{Q}\tilde{C}_0^T + \tilde{B}_0 H]\tilde{V}_0^{-1} \\ &\quad + \tfrac{1}{2} M\tilde{B}_0^T \tilde{R}\tilde{B}_0 \end{aligned} \tag{45}$$

where $\tilde{P}$ satisfies

$$\begin{aligned} 0 &= [\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0 + [\tilde{B}_0 H + (\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0)\tilde{B}_0 N] \\ &\quad \cdot \tilde{V}_0^{-1}\tilde{C}_0 + \mathcal{Q}\tilde{C}_0^T \tilde{V}_0^{-1}\tilde{C}_0]^T \tilde{P} + \tilde{P}[\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0 \\ &\quad + [\tilde{B}_0 H + (\tilde{A} + \tilde{B}_0 M_1 \tilde{C}_0)\tilde{B}_0 N]\tilde{V}_0^{-1}\tilde{C}_0 \\ &\quad + \mathcal{Q}\tilde{C}_0^T \tilde{V}_0^{-1}\tilde{C}_0] + \tilde{R}. \end{aligned} \tag{46}$$

By using (44) and (45) within a numerical search algorithm, the optimal robust reduced-order model and the scaling matrices $H$, $N$ can be determined simultaneously, thus avoiding the need to iterate between robust reduced-order model design and optimal $H$, $N$-scale evaluation.

## VI. Conclusion

Using the parameter-dependent bounding function approach developed in [2]–[5] and [7], this paper generalizes the results of [8] to systems with constant real parameter uncertainty. The design equations are presented in a concise and unified manner to facilitate their accessibility for developing numerical algorithms for practical applications. Even though efficient numerical algorithms based on homotopy methods have been developed for solving the design equations for nominal model reduction [1], the numerical tractability for the robustified design equations presented in this paper still needs to be explored. Finally, it is important to note that since only an upper bound on the $H_2$-model reduction error criterion is considered, the conservatism of the proposed robust model reduction approach will be problem dependent.

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# Computation of Stability Robustness Bounds for State-Space Models with Structured Uncertainty

Judith D. Gardiner

***Abstract*—This paper makes two contributions related to computing a bound on the size of structured real parameter perturbations under which a nominally stable matrix remains stable. First, a more efficient method is presented for computing existing bounds, requiring less time and storage than other methods. The second contribution is a means of computing tighter bounds, although at a higher computational cost.**

***Index Terms*—Algorithms, robust stability.**

## I. Introduction

This paper addresses the problem of a nominally stable linear state-space model subject to real structured perturbations which are assumed to depend linearly on a set of interval parameters. The goal is to find a bound on the size of the parameters for which the perturbed model remains stable.

This research is built largely on some recent results of Yedavalli [1]. Related work has been done by Tesi and Vicino [2], Zhou and Khargonekar [3], and Mansour [4]. This problem is closely related to the real structured singular value of $\mu$-analysis [5], [6], although the computational approach is different. An example compares the current approach to the method described in [6].

## II. Problem Statement

Let $A_0 \in \Re^{n\times n}$ be some nominal matrix which is stable in the sense that its eigenvalues all have negative real part. Define a perturbed matrix $A$ which depends linearly on a number of parameters $q_k$

$$A = A(q) = A_0 + \sum_{k=1}^{r} q_k E_k.$$

The matrices $E_k \in \Re^{n\times n}$ are known; the $q_k$'s are unknown real scalars. Matrices with polynomial or rational parameter dependence can be converted to this form, at the expense of increased size, by introducing an augmented system [6].

We wish to find a bound $\mu$ such that $A(q)$ is stable for any combination of parameters $q_k$ for which $\max_k |q_k| < \mu$. Ideally, we want the largest such $\mu$. This upper bound, call it $\mu_0$, can also be characterized as the smallest value for which some perturbed matrix $A(q)$, with $\max_k |q_k| = \mu_0$, has an eigenvalue on the imaginary axis.

There is no algorithm known for computing $\mu_0$. The problem is known to be NP-hard [7], meaning that the time for computing $\mu_0$, if it can be computed, is at least exponential in the number of parameters $r$. It is, therefore, desirable to have efficient methods for computing a lower bound on $\mu_0$, a value $\mu_* \le \mu_0$ for which it holds that $A(q)$ is stable if $\max_k |q_k| < \mu_*$.

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