Absolute Stability Criteria for Multiple Slope-Restricted Monotonic Nonlinearities

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Abstract—**Absolute stability criteria such as the classical Popov criterion guarantee stability for a class of sector-bounded nonlinearities. Although the sector restriction bounds the admissible class of the nonlinearities, the local slope of the nonlinearity may be arbitrarily large. In this paper we derive absolute stability criteria for multiple slope-restricted time-invariant monotonic nonlinearities. Like the Popov criterion, in the single-input/single-output case our results provide a simple graphical interpretation involving a straight line in a modified Popov plane.**

I. INTRODUCTION

Absolute stability theory guarantees stability of feedback systems whose forward path contains a dynamic linear time-invariant system and whose feedback path contains a memoryless (possibly time-varying) nonlinearity. These stability criteria are generally stated in terms of the linear system and apply to every element of a specified class of nonlinearities. Hence, absolute stability theory provides sufficient conditions for robust stability with a given class of uncertain elements [5], [17].

The literature on absolute stability is extensive. A convenient way to distinguish these results is to focus on the allowable class of feedback nonlinearities. Specifically, the small gain, positivity, and circle theorems guarantee stability for arbitrarily time-varying nonlinearities, whereas the Popov criterion does not. This is not surprising since the Lyapunov function upon which the small gain, positivity, and circle theorems are based is a fixed quadratic Lyapunov function which permits arbitrary time variation of the nonlinearity [5]. Alternatively, the Popov criterion is based on a Lur'e–Postnikov Lyapunov function which explicitly depends on the nonlinearity thereby restricting its allowable time variation.

Further refinements of absolute stability criteria developed in [2], [12]–[14], [19], restrict consideration to sector-bounded time-invariant nonlinearities that are monotonic or odd monotonic and are predicated on extended Lur'e-Postnikov Lyapunov functions [9], [14]. To further restrict the allowable class of feedback nonlinearities the authors in [4], [17], [22]–[24] develop absolute stability criteria by constraining the local slope of the nonlinearity. These classical absolute stability results extend the Popov criterion for sector-bounded time-invariant nonlinear functions to monotonic and odd monotonic nonlinearities by constructing stability multipliers that effectively place less restrictive conditions on the linear part of the system. However, as a result of the more involved multiplier construction, the resulting frequency domain conditions do not provide a simple graphical test as in the case of the Popov criterion.

In recent research [10], [16], [18] a new absolute stability criterion for locally slope-restricted nonlinearities involving a simple modification to the Popov multiplier was developed. Specifically, it was shown in [18] that replacing the Popov multiplier $Z(s) = 1 + Ns$ by the new multiplier $1 + Ns^{-1}$ and requiring the frequency domain condition $\mu^{-1} + (1 + Ns^{-1})G(s)$ be positive real, where $G(s)$ represents the transfer function of the linear dynamic system and μ is a bound on the local slope of the feedback nonlinearity, provides a sufficient condition for the absolute stability for systems with a monotonic time-invariant nonlinear element in the feedback path. As noted in [10], [16], however, the statement as well as the proof of the results reported in [18] were far from convincing. It should further be noted that the authors in [10] consider extensions to several differentiable nonlinearities in the feedback loop using an involved method based on integral indices along with stability inequalities that arise from the frequency domain condition. To provide connections between the proposed absolute stability condition and robust controller analysis using the parameterized Lyapunov function framework developed in [5], in this paper we extend the results of [18] to multiple slope-restricted monotonic nonlinearities as well as construct explicit Lyapunov functions along with providing the underlying Yakubovich–Kalman–Popov conditions needed to present a concise statement of these results. Specifically, an extended notion of a kinetic Lyapunov function [3] is used to show asymptotic stability of the nonlinear feedback system given by

$$\dot{x}(t) = Ax(t) - B\phi(y), \quad y(t) = Cx(t)$$

where $\phi(\cdot)$ is a time-invariant, sector-bounded memoryless nonlinearity. That is, instead of finding the condition for the state variables $x(t)$ to approach the zero equilibrium point, sufficient conditions for $\dot{x}(t)$ and the output $y(t)$ to approach zero are found. Obviously, if the system is observable and $\dot{x}(t)$ and $y(t)$ approach zero, the system arrives at one of its equilibrium points, and the two results are equivalent if the equilibrium point is unique. Finally, in the single-input/single-output (SISO) case, we show that the resulting frequency domain condition has a simple graphical interpretation involving a straight line in a modified Popov plane.

II. MATHEMATICAL PRELIMINARIES

In this section we establish definitions, notation, and several key lemmas. Let $\Re$ and C denote the real and complex numbers, let $()^T$ and $()^*$ denote transpose and complex conjugate transpose, let I_n or I denote the $n \times n$ identity matrix, and let 0_n denote the $n \times n$ zero matrix. Furthermore, $M \geq 0$ ($M > 0$) denotes the fact that the Hermitian matrix M is nonnegative (positive) definite. Let $n(s)$ and $d(s)$ be polynomials in s with real coefficients. A function $g(s)$ of the form $g(s) = n(s)/d(s)$ is called a rational function. The function $g(s)$ is called proper (respectively, strictly proper) if deg $n(s) \leq$ deg $d(s)$ (respectively, deg $n(s) <$ deg $d(s)$), where "deg" denotes the degree of the polynomial. In this paper a real-rational matrix function is a matrix whose elements are rational functions with real coefficients. Furthermore, a transfer function $G(s)$ is called proper (respectively, strictly proper) if every element of $G(s)$ is proper (respectively, strictly proper). Finally, an asymptotically stable transfer function is a transfer function each of whose poles is in the open left-half plane. The space of asymptotically stable transfer functions is denoted by $\mathcal{RH}_\infty$, i.e., the real-rational subset of $\mathcal{H}_\infty$. Let

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

denote a state space realization of a transfer function $G(s)$, that is, $G(s) = C(sI - A)^{-1}B + D$. The notation $\overset{\min}{\sim}$ is used to denote a minimal realization. In addition, the parahermitian conjugate $G^{\sim}(s)$

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of $G(s)$ has the realization

$$G^{\sim}(s) \sim \left[\begin{array}{c|c} -A^T & C^T \\ \hline -B^T & D^T \end{array}\right].$$

Furthermore, the Hermitian part of G is given by He $G = \frac{1}{2}\,(G+G^*)$.

A square transfer function $G(s)$ is called positive real [1, p. 216] if 1) all poles of $G(s)$ lie in the closed left-half plane, and 2) He $G(s)$ is nonnegative definite for Re $s > 0$. A square transfer function $G(s)$ is called strictly positive real [21] if 1) $G(s)$ is asymptotically stable and 2) He $G(\jmath\omega)$is positive definite for all real ω. Recall that a minimal realization of a positive real transfer function is stable in the sense of Lyapunov, while a minimal realization of a strictly positive real transfer function is asymptotically stable.

For notational convenience we will omit all matrix dimensions throughout the paper and assume that all quantities have compatible dimensions. Furthermore, in this paper, $G(s)$ will denote an $m \times m$ transfer function with input $u \in \Re^m$, output $y \in \Re^m$, and internal state $x \in \Re^n$. Next, we state the strict positive real lemma used to characterize strict positive realness in the state-space setting.

Lemma 2.1 (Strict Positive Real Lemma [20]):

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

is strictly positive real if and only if there exist matrices P, L, and W with P positive definite such that

$$0 = A^TP + PA + L^TL, \tag{2.1}$$

$$0 = B^TP - C + W^TL, \tag{2.2}$$

$$0 = D + D^T - W^TW \tag{2.3}$$

are satisfied, the pair (A, L) is observable, and rank $\hat{G}(\jmath\omega) = m$, $\omega \in \Re$, where

$$\hat{G}(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline L & W \end{array}\right].$$

Finally, we state a key lemma involving controllability of an augmented pair.

Lemma 2.2 [10]: Given a triple (A, B, C), if (A, B) is controllable and det $A \neq 0$, then

$$\left(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}\right)$$

is controllable if and only if det $CA^{-1}B \neq 0$.

III. ABSOLUTE STABILITY CRITERION FOR MULTIPLE SLOPE-RESTRICTED NONLINEARITIES

In this section we consider the absolute stability problem for a class Φ of locally slope restricted monotonic time-invariant nonlinearities $\phi : \Re^m \to \Re^m$. Specifically, given

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

we derive conditions that guarantee global asymptotic stability of the negative feedback interconnection of $G(s)$ and ϕ for all $\phi \in \Phi$. Note that the negative feedback interconnection of $G(s)$ and $\phi(\cdot)$ has the state-space representation

$$\dot{x}(t) = Ax(t) - B\phi(y), \tag{3.1}$$

$$y(t) = Cx(t). \tag{3.2}$$

To state our main result, the following definitions are needed. Let $\mu \in \Re^{m\times m}$ be a positive definite diagonal matrix. Next, define the set Φ of allowable nonlinearities ϕ by

$$\begin{aligned} \Phi \triangleq \{\phi : \Re^m \to \Re^m : \phi(y) = [\phi_1(y_1), \cdots, \phi_m(y_m)]^T, \\ \phi(\cdot) \text{ is differentiable}, \\ 0 < \phi_i'(y_i) < \mu_i, \quad i = 1, \cdots, m, \; y \in \Re^m\}. \end{aligned} \tag{3.3}$$

Note that the nonlinear functions considered, $\phi \in \Phi$, have decoupled components but unlike the multivariable extensions of the Popov criterion [5], [11] we assume a local slope constraint on the nonlinearities. In the scalar case, $m = 1$, ϕ satisfies the usual local slope condition $0 < \phi'(y) < \mu$, $y \in \Re$.

For the statement of the main result define

$$A_a \triangleq \begin{bmatrix} A & 0_{n\times m} \\ C & 0_m \end{bmatrix}, \quad B_a \triangleq \begin{bmatrix} B \\ 0_m \end{bmatrix},$$

$$C_a \triangleq [0_{m\times n}\; I_m], \quad S \triangleq [C\, 0_m].$$

Theorem 3.1: Let

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

be symptotically stable, let $N \triangleq$ diag $[N_1, N_2, \cdots, N_m]$ be nonnegative-definite, assume det $CA^{-1}B \neq 0$, and define

$$Z(s) \triangleq I + Ns^{-1}. \tag{3.4}$$

Then

$$\mathcal{G}(s) \triangleq \mu^{-1} + Z(s)G(s) \tag{3.5}$$

is strictly positive real if and only if there exist matrices P, L, and W with P positive definite satisfying

$$0 = A_a^TP + PA_a + L^TL, \tag{3.6}$$

$$0 = B_a^TP - NC_a - S + W^TL, \tag{3.7}$$

$$0 = 2\mu^{-1} - W^TW. \tag{3.8}$$

In this case

$$V(\dot{x}, y) = \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix}^T P \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} + 2\sum_{i=1}^{m} \int_0^{y_i} N_i\phi_i'(\sigma)\sigma\, d\sigma \tag{3.9}$$

where $y(t) = Cx(t)$, is a Lyapunov function that guarantees that the negative feedback interconnection of $G(s)$ and $\phi(\cdot)$ is globally asymptotically stable for all $\phi \in \Phi$.

Proof: First, define $z(t) \triangleq \dot{x}(t)$ so that

$$\dot{z}(t) = Az(t) - B\phi'(y)\dot{y}, \tag{3.10}$$

$$\dot{y}(t) = Cz(t) \tag{3.11}$$

and

$$\dot{x}_a(t) = A_a x_a(t) - B_a f(y) \tag{3.12}$$

where

$$x_a(t) \triangleq \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix}, \quad f(y) \triangleq \phi'(y)\dot{y}(t).$$

Furthermore, note that in this case $y(t) = C_a x_a(t)$ and $\dot{y}(t) = Sx_a(t)$. Now, since (A, B, C) is minimal, (A, B) is controllable. Hence, it follows from Lemma 2.2 that if det $CA^{-1}B \neq 0$ then (A_a, B_a) is also controllable.

Next, we show that (3.6)–(3.8) imply that $\mathcal{G}(s)$ is strictly positive real. To do this, add and subtract $j\omega P$ to and from (3.6) to obtain

$$0 = (-j\omega I - A_a)^T P + P(j\omega I - A_a) - L^T L. \tag{3.13}$$

Now, forming $B_a^T(-j\omega I - A_a)^{-T}$ (3.13) $(j\omega I - A_a)^{-1} B_a$ and using (3.7) we obtain

$$\begin{aligned}&[NC_a + S - W^T L](j\omega I - A_a)^{-1} B_a \\ &\quad + B_a^T(-j\omega I - A_a)^{-T}[NC_a + S - W^T L]^T \\ &\quad = B_a^T(-j\omega I - A_a)^{-T} L^T L(j\omega I - A_a)^{-1} B_a. \end{aligned} \tag{3.14}$$

Adding and subtracting $W^T W$ to and from (3.14), using (3.8), and grouping terms yields

$$\begin{aligned}&[NC_a + S](j\omega I - A_a)^{-1} B_a \\ &\quad + B_a^T(-j\omega I - A_a)^{-T}[NC_a + S]^T + 2\mu^{-1} \\ &\quad = [W + L(j\omega I - A_a)^{-1} B_a]^* \\ &\quad \cdot [W + L(j\omega I - A_a)^{-1} B_a]. \end{aligned} \tag{3.15}$$

Next, using the identities

$$\begin{aligned}(j\omega I - A_a)^{-1} &= \begin{bmatrix} (j\omega I - A) & 0 \\ -C & j\omega I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (j\omega I - A)^{-1} & 0 \\ (j\omega)^{-1} C(j\omega I - A)^{-1} & (j\omega)^{-1} I \end{bmatrix}\end{aligned}$$

$$\begin{aligned}S(j\omega I - A_a)^{-1} B_a &= G(j\omega), \\ C_a(j\omega I - A_a)^{-1} B_a &= \frac{G(j\omega)}{j\omega}.\end{aligned}$$

it follows from (3.15) and the rank condition in ii) of Lemma 2.1 that He $\mathcal{G}(j\omega) > 0$. Hence $\mathcal{G}(s)$ is strictly positive real.

Conversely, assuming that $\mathcal{G}(s)$ is strictly positive real, spectral factorization theory guarantees the existence of a spectral factor $\mathcal{N}(s)$ such that $\mathcal{G}(s) + \mathcal{G}^{\sim}(s) = \mathcal{N}^{\sim}(s)\mathcal{N}(s)$, where $\mathcal{N}^{\pm 1}(s) \in \mathcal{RH}_\infty$. The existence of P, L, and W with P positive definite satisfying (3.6)–(3.8) now follows from standard algebraic state-space realization manipulations.

Alternatively, the result follows from a direct consequence of Lemma 2.1 by noting that $\mathcal{G}(s)$ has a minimal realization given by

$$\mathcal{G}(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A_a & B_a \\ \hline NC_a + S & \mu^{-1} \end{array}\right].$$

Next, for $\phi \in \Phi$ consider the Lyapunov function candidate (3.9). First note that using integration by parts the integral term in (3.9) is equivalent to

$$\sum_{i=1}^{m} \int_0^{y_i} N_i \phi_i'(\sigma)\sigma d\sigma = \sum_{i=1}^{m} \left\{ N_i \phi_i(y_i) y_i - \int_0^{y_i} N_i \phi_i(\sigma) d\sigma \right\}.$$

Now, since $\phi'(y) > 0$, for all $\phi \in \Phi$, it follows that $\sum_{i=1}^{m} \{N_i \phi_i(y_i) y_i - \int_0^{y_i} N_i \phi_i(\sigma)\, d\sigma\} > 0$. Furthermore, since P is positive definite, it follows that $V(x_a)$ is positive definite. The corresponding Lyapunov derivative is given by

$$\begin{aligned}\dot{V}(x_a) &= x_a^T[A_a^T P + PA_a]x_a \\ &\quad - f^T(y)[B_a^T P - NC_a]x_a \\ &\quad - x_a^T[B_a^T P - NC_a]^T f(y). \end{aligned} \tag{3.16}$$

Next, it follows from (3.3) that $\phi'(y)(I - \mu^{-1}\phi'(y)) > 0$, for all $\phi \in \Phi$. Hence, $2\, \dot{y}^T \phi'(y)(I - \mu^{-1}\phi'(y))\dot{y} \geq 0$. Now, adding and subtracting $2\, \dot{y}^T \phi'(y)(I - \mu^{-1}\phi'(y))\dot{y}$ to and from (3.16) and grouping terms yields

$$\begin{aligned}\dot{V}(x_a) &= x_a^T[A_a^T P + PA_a]x_a \\ &\quad - f^T(y)[B_a^T P - NC_a - S]x_a \\ &\quad - x_a^T[B_a^T P - NC_a - S]^T f(y) \\ &\quad - 2f^T(y)\mu^{-1} f(y) - 2\dot{y}^T \phi'(y) \\ &\quad \cdot (I - \mu^{-1}\phi'(y))\dot{y} \end{aligned} \tag{3.17}$$

or, using (3.6)–(3.8), yields

$$\begin{aligned}\dot{V}(x_a) &= -[Lx_a - Wf(y)]^T[Lx_a - Wf(y)] \\ &\quad - 2\dot{y}^T \phi'(y)(I - \mu^{-1}\phi'(y))\dot{y}. \end{aligned} \tag{3.18}$$

Since $2\dot{y}^T \phi'(y)(I - \mu^{-1}\phi'(y))\dot{y} \geq 0$, it follows that $\dot{V}(x_a) \leq 0$, which proves stability in the sense of Lyapunov.

To show global asymptotic stability we need to show that $\dot{V}(x_a) = 0$ implies that $x = 0$. Note that $\dot{V}(x_a) = 0$ implies that $\dot{y}(t) = 0$, $t \geq 0$, and hence $f(y) = 0$ and $Lx_a(t) = 0$. Furthermore, in this case $\dot{x}_a(t) = A_a x_a(t) - B_a f(y) = A_a x_a(t)$. Thus, using $\dot{x}_a(t) = A_a x_a(t)$, $Lx_a(t) = 0$, and the observability of (A_a, L), it follows from the PBH test that $x_a(t) = 0$, $t \geq 0$, which further implies, since (A, C) is observable, that $x(t) = 0$, $t \geq 0$. Thus, the only solution satisfying $\dot{V}(x, y) = 0$ is the $x(t) = 0$, $t \geq 0$, solution and hence it follows from the LaSalle's theorem [20] that global asymptotic stability holds. □

Theorem 3.1 presents a generalization of Theorem 1 of [18] to the case of multivariable plants containing an arbitrary number of memoryless time-invariant slope-restricted monotonic nonlinearities.

The form of $\mathcal{G}(s)$ given by (3.5) is standard in the classical absolute stability theory [14] in which $Z(s)$ is a stability multiplier that distinguishes the class of the allowable feedback nonlinearities. As mentioned in the Introduction specific cases include memoryless time-invariant nonlinearities [15], monotonic and odd monotonic nonlinearities [9], [12]–[14], and locally slope restricted nonlinearities [4], [22]–[24]. A key difference between the results in Theorem 3.1 and the classical theory on monotonic and odd monotonic nonlinearities [9], [12]–[14] is that the multiplier $Z(s)$ in (3.4) involves a simple twist on the Popov multiplier in contrast to the more involved positive real multipliers involving partial fraction expansions of driving point impedances of resistor-inductor ($\mathcal{RL}$) and resistor-capacitor ($\mathcal{RC}$) combinations which exhibit interlacing pole-zero patterns on the negative real axis [9], [12], [13] and non-positive real plant-dependent multipliers [4], [7], [23].

In the SISO case, the frequency domain condition in Theorem 3.1 has an interesting geometric interpretation. Specifically, setting $G(j\omega) = x + jy$, He $\mathcal{G}(j\omega) > 0$ is equivalent to

$$\frac{1}{\mu} + x + \frac{N}{\omega} y > 0. \tag{3.19}$$

Condition (3.19) is a frequency domain stability criterion with a graphical interpretation in a modified Popov plane, involving Re G and ω^{-1} Im G, in terms of a straight line with a real axis-intercept $-1/\mu$ and slope $-1/N$.

Since the condition presented in Theorem 3.1 is only sufficient for absolute stability a natural question that arises is for what class of systems will this criterion give less conservative predictions over the classical Popov criterion. In order to address this question first recall that the effect that the Popov multiplier $1 + Ns$ has on the Nyquist plot is to rotate each point on the Nyquist plot in the counter clockwise direction. Hence, if the Nyquist plot of the plant transfer function $G(s)$ enters the second quadrant then it is clear that there does not exist an N such that $(1 + Ns)\, G(s)$ is positive real. Thus, in

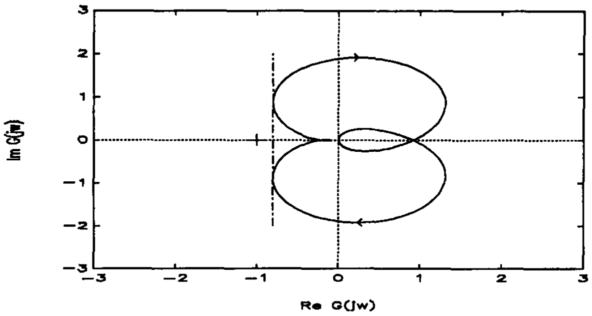

Fig. 1. Positive real analysis.

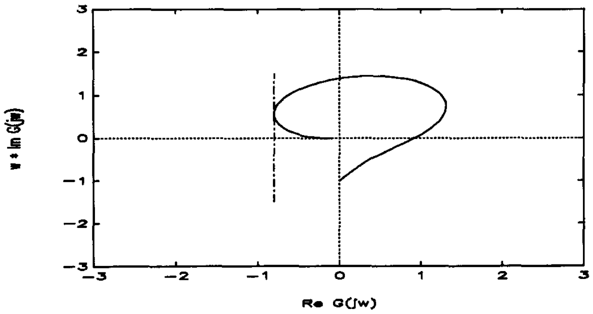

Fig. 2. Popov analysis.

this case, the Popov criterion does not provide any improvement over the positive real test (see the example in Section IV). Alternatively, since the effect of the proposed multiplier $1 + Ns^{-1}$ is to rotate each point on the Nyquist plot in the clockwise direction, the criterion in Theorem 3.1 will always give less conservative predictions over the Popov criterion when the Nyquist plot of $G(s)$ resides in the first and second quadrants. For example, since the Nyquist plot of the class of third-order transfer functions given by

$$G(s) = \frac{s^2}{a_0 s^3 + a_1 s^2 + a_2 s + a_3}$$

where $a_0 > 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, and $a_2 a_1 > a_0 a_3$ will always enter the second quadrant, the proposed criterion would give less conservative predictions over the Popov criterion. Of course, using similar arguments as above, if the Nyquist plot of $G(s)$ resides in the third and fourth quadrants then the proposed criterion would not give any improvement over the positivity criterion while the Popov criterion would give less conservative predictions. Hence, the utility of the proposed criterion is when the Popov criterion fails. Finally, it should be noted that the more general class of multipliers consisting of the $\mathcal{RL}$ and the $\mathcal{RC}$ class [9], [12], [13] place less restrictive conditions on $G(s)$ and hence allow the Nyquist plot to reside in all four quadrants. As a result of the more involved multiplier construction, however, the resulting frequency domain conditions provide a complex graphical interpretation involving frequency-dependent off-axis circles in the Nyquist plane [9]. Alternatively, if the classical off-axis circle criterion is used where a single bounding circle in the Nyquist plane is employed [14] as opposed to a family of frequency dependent circles then conservatism will be introduced in the stability predictions.

Remark 3.1: Note that the class Φ of nonlinearities becomes larger as μ increases. In fact, as μ increases the strict positive real condition (3.5) becomes more difficult to satisfy, as expected. Furthermore,

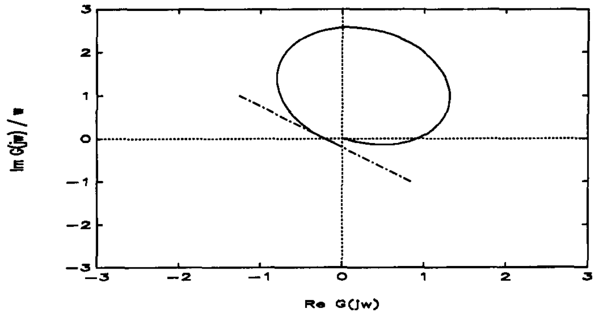

Fig. 3. Modified Popov analysis.

even though the frequency domain condition in Theorem 3.1 does not involve an explicit sector constraint on the nonlinearities $\phi(y)$, the requirement that $\phi \in \Phi$ implies that $0 < \phi_i(y_i)y_i < \mu_i y_i^2$, $y_i \in \Re$, $i = 1, \cdots, m$.

Next, we partially relax the assumption det $CA^{-1}B \neq 0$ and provide an alternative Lyapunov function construction for the absolute stability criterion given in Theorem 3.1. The following result does not require a system augmentation of the form (3.12), however, in this case we assume that every element of the stable transfer function $G(s)$ has at least one zero at the origin, i.e., $G(0) = 0$.

Theorem 3.2: Let

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

be asymptotically stable, let $N \triangleq \text{diag}\,[N_1, N_2, \cdots, N_m]$ be nonnegative-definite, assume that $G(0) = 0$, and define

$$Z(s) \triangleq I + Ns^{-1}. \tag{3.20}$$

Then

$$\mathcal{G}(s) \triangleq \mu^{-1} + Z(s)G(s) \tag{3.21}$$

is strictly positive real, if and only if there exists matrices P, L, and W with P positive definite satisfying

$$0 = A^T P + PA + L^T L, \tag{3.22}$$

$$0 = B^T P - NCA^{-1} - C + W^T L, \tag{3.23}$$

$$0 = 2\mu^{-1} - W^T W. \tag{3.24}$$

In this case

$$V(x) = x^T P x + 2\sum_{i=1}^{m} \int_0^{y_i} N_i \phi_i'(\sigma)\sigma \, d\sigma \tag{3.25}$$

where $y(t) = Cx(t)$, is a Lyapunov function that guarantees that the negative feedback interconnection of $G(s)$ and ϕ is globally asymptotically stable for all $\phi \in \Phi$.

Proof: The proof is similar to the proof of Theorem 3.1. ☐

Remark 3.2: Note that since $G(s)$ is asymptotically stable A^{-1} exists. Furthermore, in order to construct the proof of Theorem 3.2, it is helpful to note that since every element of the stable transfer function $G(s)$ has at least one zero at the origin, $CA^{-1}B = 0$ and $CA^{-1}(sI - A)^{-1}B = \frac{G(s)}{s}$.

IV. ILLUSTRATIVE NUMERICAL EXAMPLE

For illustrative purposes we consider the stable linear system with transfer function

$$G(s) = \frac{s^2 - 0.2s - 0.1}{s^3 + 2s^2 + s + 1}. \tag{4.1}$$

The closed-loop system (3.1) with linear uncertainty $\phi(y) = Fy$ is asymptotically stable for $0 \leq F \leq 4.6$. The Nyquist plot for the linear system is shown in Fig. 1. Hence, it follows from the positivity theorem that the linear system is asymptotically stable for all time-invariant monotone nonlinearities in the sector [0, 1.24). Fig. 2 shows the corresponding Popov plot which, in this case, gives a Popov sector of [0, 1.24). Hence, since the Nyquist plot of $G(s)$ enters the second quadrant the Popov criterion does not provide any improvement over the positive real test. Finally, using Theorem 3.1 we construct a modified Popov plot shown in Fig. 3. Using the modified Popov exclusionary half plane graphical test given by (3.19), the absolute stability sector is now found to be [0,4.6), which is a significant improvement over the positivity and Popov sectors for time-invariant monotone feedback nonlinearities.

V. CONCLUSION

In this paper we extended the SISO absolute stability criterion for locally slope-restricted monotonic nonlinearities developed in [18] to multivariable systems containing an arbitrary number of monotonic slope bounded nonlinearities. Specifically, explicit Lyapunov functions along with extended Yakubovich–Kalman–Popov conditions are given. These results can be used to synthesize robust feedback controllers in the spirit of [6], [8], [9].

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Simultaneous Disturbance Rejection and Regular Row by Row Decoupling With Stability: A Geometric Approach

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***Abstract*—The simultaneous disturbance rejection problem and regular row by row decoupling problem with stability is solved here through a geometric approach. It is shown in this paper that the combined problem has a solution if and only if each problem, separately, has a solution.**

I. INTRODUCTION

As far as its geometric setting is concerned, the combined problem of disturbance rejection and input–output decoupling for linear time-invariant systems, by static state feedback, has been first discussed in [1] and [3], almost 20 years ago. This problem has been recently revisited in [9], while the structural approach has been used in [2] in order to obtain easy-to-verify solvability conditions.

As the main result of the research referenced above, a nice property has been found. Indeed, when no stability constraint is imposed, it

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