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Robust Reduced-Order Modeling Via the Optimal Projection Equations with Petersen–Hollot Bounds

WASSIM M. HADDAD AND DENNIS S. BERNSTEIN

Abstract—**An optimal reduced-order modeling problem with parametric plant uncertainty is considered. A model-reduction bound suggested by recent work of Petersen and Hollot is utilized for guaranteeing robust reduced-order modeling over a specified range of uncertain parameters. Necessary conditions which generalize the optimal projection equations for model reduction are used to characterize the reduced-order model which minimizes the model-reduction bound. The optimality equations thus effectively serve as sufficient conditions for characterizing robust reduced-order models.**

I. Introduction

It has been shown in [1]–[3] that the first-order necessary conditions for quadratically optimal reduced-order modeling, estimation, and control can be transformed into coupled systems of two, three, and four matrix equations, respectively. This coupling is due to an oblique projection which arises as a direct consequence of optimality. In a series of papers [4]–[6] the optimal projection approach was generalized to the problems of robust reduced-order estimation and control in the presence of real-valued, structured parameter uncertainty. This was accomplished by incorporating the quadratic uncertainty bound proposed in [7] within the optimal projection framework.

Manuscript received August 4, 1987; revised October 16, 1987. This work was supported in part by the Air Force Office of Scientific Research under Contract F49620-86-C-0002.

W. M. Haddad is with the Department of Mechanical Engineering, Florida Institute of Technology, Melbourne, FL 32901.

D. S. Bernstein is with the Government Aerospace Systems Division, Harris Corporation, Melbourne, FL 32902.

IEEE Log Number 8821012.

The purpose of the present note is to complete this cycle of results by similarly extending the results of [1]. Our goal is thus to obtain *robust* reduced-order models over a specified range of parametric plant uncertainty. As in [4]–[6], the main idea is to bound the effect of the uncertain parameters on the model-reduction error over the uncertainty range and then determine a reduced-order model which minimizes the model-reduction bound. The resulting generalization of the optimal projection equations now serves as a *sufficient* condition for robust model reduction by virtue of the fact that a *bound* on the model-reduction error is being minimized rather than the model-reduction error itself. These optimality conditions now comprise a coupled system of three algebraic matrix equations which reduce to the result of [1] when the uncertainty bounds are absent.

II. Notation and Definitions

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$	real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$
I_r, $(\ )^T$, $\mathbb{E}$	$r \times r$ identity matrix, transpose, expected value
$\oplus$	Kronecker sum
$\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$	$r \times r$ symmetric, nonnegative-definite, positive-definite matrices
$Z_1 \le Z_2$, $Z_1 < Z_2$	$Z_2 - Z_1 \in \mathbb{N}^r$, $Z_2 - Z_1 \in \mathbb{P}^r$, $Z_1, Z_2 \in \mathbb{S}^r$
$n, l, n_m, m; \tilde{n}$	positive integers; $n + n_m$
$x, y, y_m, x_m, \tilde{x}$	$n, l, l, n_m, \tilde{n}$-dimensional vectors
$A, \Delta A; B, C$	$n \times n$ matrices; $n \times m$, $l \times n$ matrices
A_m, B_m, C_m	$n_m \times n_m$, $n_m \times m$, $l \times n_m$ matrices
$\tilde{A}, \Delta\tilde{A}, \tilde{B}$	$\begin{bmatrix} A & 0 \\ 0 & A_m \end{bmatrix}, \begin{bmatrix} \Delta A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} B \\ B_m \end{bmatrix}$
R	model-reduction error-weighting matrix in $\mathbb{P}^l$
$w(\cdot)$	m-dimensional white noise
V	intensity of $w(\cdot)$ in $\mathbb{P}^m$
$\tilde{R}$	$\begin{bmatrix} C^TRC & -C^TRC_m \\ -C_m^TRC & C_m^TRC_m \end{bmatrix}$,
$\tilde{V}$	$\begin{bmatrix} BVB^T & BVB_m^T \\ B_mVB^T & B_mVB_m^T \end{bmatrix}$.

III. Robust Model-Reduction Problem

Let $\mathcal{U} \subset \mathbb{R}^{n\times n}$ denote the set of uncertain perturbations ΔA of the nominal plant matrix A.

Robust Model-Reduction Problem: For fixed $n_m \le n$, determine (A_m, B_m, C_m) such that, for the system consisting of the nth-order disturbed plant

$$\dot{x}(t) = (A + \Delta A)x(t) + Bw(t), \qquad t \in [0, \infty), \tag{3.1}$$

outputs

$$y(t) = Cx(t), \tag{3.2}$$

and n_mth-order model

$$\dot{x}_m(t) = A_mx_m(t) + B_mw(t), \tag{3.3}$$

$$y_m(t) = C_mx_m(t), \tag{3.4}$$

the model-reduction criterion

$$J(A_m, B_m, C_m) \triangleq \sup_{\Delta A \in \mathcal{U}} \limsup_{t\to\infty} \mathbb{E}[y(t) - y_m(t)]^T R [y(t) - y_m(t)] \tag{3.5}$$

is minimized.

For each reduced-order model (A_m, B_m, C_m) and system variation $\Delta A \in \mathfrak{U}$, the augmented system (3.1)-(3.4) is given by

$$\dot{\tilde{x}}(t)=(\tilde{A}+\Delta\tilde{A})\tilde{x}(t)+\tilde{B}w(t), \qquad t \in [0, \infty) \tag{3.6}$$

where $\tilde{x}(t) \triangleq [x^T(t), x_m^T(t)]^T$.

IV. Sufficient Conditions for Robust Performance

The following result is immediate.

Lemma 4.1: Suppose A_m is asymptotically stable and $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathfrak{U}$. Then

$$J(A_m, B_m, C_m)= \sup_{\Delta A\in\mathfrak{U}} \operatorname{tr} \tilde{Q}_{\Delta\tilde{A}}\tilde{R} \tag{4.1}$$

where $\tilde{Q}_{\Delta\tilde{A}} \triangleq \lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)\tilde{x}(t)^T] \in \mathbb{N}^{\tilde{n}}$ is the unique solution to

$$0=(\tilde{A}+\Delta\tilde{A})\tilde{Q}_{\Delta\tilde{A}}+\tilde{Q}_{\Delta\tilde{A}}(\tilde{A}+\Delta\tilde{A})^T+\tilde{V}. \tag{4.2}$$

We now determine an upper bound for J given by (4.1).

Theorem 4.1: Let $\Omega:\mathbb{N}^{\tilde{n}} \to \mathbb{S}^{\tilde{n}}$ be such that

$$\Delta\tilde{A}\mathfrak{Q}+\mathfrak{Q}\Delta\tilde{A}^T\leq\Omega(\mathfrak{Q}),\ \Delta A \in \mathfrak{U},\ \mathfrak{Q} \in \mathbb{N}^{\tilde{n}} \tag{4.3}$$

and, for given (A_m, B_m, C_m), suppose there exists $\mathfrak{Q} \in \mathbb{N}^{\tilde{n}}$ satisfying

$$0=\tilde{A}\mathfrak{Q}+\mathfrak{Q}\tilde{A}^T+\Omega(\mathfrak{Q})+\tilde{V} \tag{4.4}$$

and suppose the pair $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is stabilizable for all $\Delta A \in \mathfrak{U}$. Then A_m is asymptotically stable, $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathfrak{U}$,

$$\tilde{Q}_{\Delta\tilde{A}}\leq\mathfrak{Q},\ \Delta A \in \mathfrak{U}, \tag{4.5}$$

where $\tilde{Q}_{\Delta\tilde{A}}$ satisfies (4.2), and

$$J(A_m, B_m, C_m)\leq\operatorname{tr} \mathfrak{Q}\tilde{R}. \tag{4.6}$$

Proof: See [5]. □

Remark 4.1: Theorem 4.1 provides sufficient conditions for reduced-order modeling with an upper bound on modeling error. The result yields, in addition, the result that A_m and $A + \Delta A$ are asymptotically stable. Thus, it is important to emphasize that our results are effectively limited to systems which remain asymptotically stable over the class of uncertainties. Relevant applications include, for example, damped flexible structures with uncertain modal data.

V. Uncertainty Structure and the Petersen-Hollot Bound

The uncertainty set $\mathfrak{U}$ is assumed to be of the form

$$\mathfrak{U}=\left\{\Delta A \in \mathbb{R}^{n\times n} : \Delta A=\sum_{i=1}^{p} D_iM_iN_iE_i,\ M_iM_i^T\leq\bar{M}_i,\ N_i^TN_i\leq\bar{N}_i,\ i=1, \cdots, p\right\} \tag{5.1}$$

where, for $i = 1, \cdots, p$: $D_i \in \mathbb{R}^{n\times r_i}$ and $E_i \in \mathbb{R}^{t_i\times n}$ are fixed matrices denoting the structure of the uncertainty; $\bar{M}_i \in \mathbb{N}^{r_i}$ and $\bar{N}_i \in \mathbb{N}^{t_i}$ are given uncertainty bounds; and $M_i \in \mathbb{R}^{r_i\times s_i}$, $N_i \in \mathbb{R}^{s_i\times t_i}$ are uncertain matrices. The augmented system thus has structured uncertainty of the form

$$\Delta\tilde{A}=\sum_{i=1}^{p} \tilde{D}_iM_iN_i\tilde{E}_i \tag{5.2}$$

where

$$\tilde{D}_i \triangleq \begin{bmatrix} D_i \\ 0 \end{bmatrix},\quad \tilde{E}_i \triangleq [E_i \quad 0]. \tag{5.3}$$

We now specify the function Ω satisfying (4.3).

Proposition 5.1: The function

$$\Omega(\mathfrak{Q}) \triangleq \sum_{i=1}^{p} \tilde{D}_i\bar{M}_i\tilde{D}_i^T+\mathfrak{Q}\tilde{E}_i^T\bar{N}_i\tilde{E}_i\mathfrak{Q} \tag{5.4}$$

satisfies (4.3) with $\mathfrak{U}$ given by (5.1).

Proof: For $i = 1, \cdots, p$,

$$\begin{aligned}0&\leq[\tilde{D}_iM_i-\mathfrak{Q}\tilde{E}_i^TN_i^T][\tilde{D}_iM_i-\mathfrak{Q}\tilde{E}_i^TN_i^T]^T\\ &=\tilde{D}_iM_iM_i^T\tilde{D}_i^T+\mathfrak{Q}\tilde{E}_i^TN_i^TN_i\tilde{E}_i\mathfrak{Q}-(\tilde{D}_iM_iN_i\tilde{E}_i\mathfrak{Q}+\mathfrak{Q}\tilde{E}_i^TN_i^TM_i^T\tilde{D}_i^T)\\ &\leq\tilde{D}_i\bar{M}_i\tilde{D}_i^T+\mathfrak{Q}\tilde{E}_i^T\bar{N}_i\tilde{E}_i\mathfrak{Q}-(\tilde{D}_iM_iN_i\tilde{E}_i\mathfrak{Q}+\mathfrak{Q}\tilde{E}_i^TN_i^TM_i^T\tilde{D}_i^T).\end{aligned}$$

Summing over i yields (4.3). □

Remark 5.1: The bound (5.4) was originally used in [7] for unit-rank perturbations with scalar uncertain parameters. For further details, see [4]-[6].

VI. The Auxiliary Minimization Problem

Our goal is to minimize the error bound (4.6).

Auxiliary Minimization Problem: Determine $(\mathfrak{Q}, A_m, B_m, C_m)$ with $\mathfrak{Q} \in \mathbb{N}^{\tilde{n}}$ which minimize

$$\mathfrak{J}(\mathfrak{Q}, A_m, B_m, C_m) \triangleq \operatorname{tr} \mathfrak{Q}\tilde{R} \tag{6.1}$$

subject to

$$0=\tilde{A}\mathfrak{Q}+\mathfrak{Q}\tilde{A}^T+\sum_{i=1}^{p} [\tilde{D}_i\bar{M}_i\tilde{D}_i^T+\mathfrak{Q}\tilde{E}_i^T\bar{N}_i\tilde{E}_i\mathfrak{Q}]+\tilde{V} \tag{6.2}$$

and

$$(\tilde{V}^{1/2}, \tilde{A}+\Delta\tilde{A}) \text{ is stabilizable, } \Delta A \in \mathfrak{U}. \tag{6.3}$$

Proposition 6.1: If $(\mathfrak{Q}, A_m, B_m, C_m)$ satisfy (6.2) and (6.3) with $\mathfrak{Q} \geq 0$, then A_m is asymptotically stable, $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathfrak{U}$, and

$$J(A_m, B_m, C_m)\leq\mathfrak{J}(\mathfrak{Q}, A_m, B_m, C_m). \tag{6.4}$$

Proof: With Ω given by (5.4), Proposition 5.1 implies that (4.3) is satisfied. Hence, with (6.3), the hypotheses of Theorem 4.1 are satisfied so that the system (3.6) is stable over $\mathfrak{U}$ with model-reduction bound (4.6). Note that with (6.1), (6.4) is merely a restatement of (4.6). □

VII. Necessary Conditions for the Auxiliary Minimization Problem

Rigorous application of the Lagrange multiplier technique requires additional technical assumptions. Specifically, we further restrict $(\mathfrak{Q}, A_m, B_m, C_m)$ to the open set

$$\mathcal{S} \triangleq \{(\mathfrak{Q}, A_m, B_m, C_m) : \mathfrak{Q} \in \mathbb{P}^{\tilde{n}},\ \tilde{\mathfrak{A}} \text{ is asymptotically stable, and } (A_m, B_m, C_m) \text{ is controllable and observable}\}$$

where

$$\tilde{\mathfrak{A}} \triangleq \left(\tilde{A}+\sum_{i=1}^{p} \tilde{E}_i^T\bar{N}_i\tilde{E}_i\mathfrak{Q}\right) \oplus \left(\tilde{A}+\sum_{i=1}^{p} \tilde{E}_i^T\bar{N}_i\tilde{E}_i\mathfrak{Q}\right).$$

Remark 7.1: The constraint $(\mathfrak{Q}, A_m, B_m, C_m) \in \mathcal{S}$ is not required for robust reduced-order modeling since, as shown by Proposition 6.1, only (6.2) and (6.3) are required. As will be seen from the proof of Theorem 7.1, the set $\mathcal{S}$ constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the auxiliary minimization problem. Specifically, $\mathfrak{Q} > 0$ replaces $\mathfrak{Q} \geq 0$ by an open set constraint, asymptotic stability of $\tilde{\mathfrak{A}}$ serves as a normality condition which further implies that the dual $\mathcal{P}$ of $\mathfrak{Q}$ is nonnegative definite, and (A_m, B_m, C_m) minimal is a nondegeneracy condition.

The following factorization lemma is needed for the statement of the main result. For details, see [1].

Lemma 7.1: If $\hat{Q}, \hat{P} \in \mathbb{N}^n$ and rank $\hat{Q}\hat{P} = n_m$, then there exist $n_m \times n$ G, Γ, and $n_m \times n_m$ invertible M such that

$$\hat{Q}\hat{P} = G^T M \Gamma, \tag{7.1}$$

$$\Gamma G^T = I_{n_m}. \tag{7.2}$$

Furthermore, G, M, and Γ are unique except for a change of basis in $\mathbb{R}^{n_m}$.

Recall from [1] that

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} = G^T\Gamma \tag{7.3}$$

is an oblique projection, where $(\ )^{\#}$ denotes group generalized inverse. Define the complementary projection $\tau_\perp \triangleq I_n - \tau$ and call (G, M, Γ) satisfying (7.1), (7.2) a *projective factorization* of $\hat{Q}\hat{P}$. Furthermore, define the notation

$$D \triangleq \sum_{i=1}^{p} D_i \bar{M}_i D_i^T, \quad E \triangleq \sum_{i=1}^{p} E_i^T \bar{N}_i E_i.$$

Theorem 7.1: Suppose $(\mathcal{Q}, A_m, B_m, C_m) \in \mathcal{S}$ solves the auxiliary minimization problem with $\mathcal{U}$ given by (5.1). Then there exist $Q, \hat{Q}, \hat{P} \in \mathbb{N}^n$ such that $\mathcal{Q}, A_m, B_m, C_m$ are given by

$$\mathcal{Q} = \begin{bmatrix} Q+\hat{Q} & \hat{Q}\Gamma^T \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^T \end{bmatrix}, \tag{7.4}$$

$$A_m = \Gamma(A + QE)G^T, \tag{7.5}$$

$$B_m = \Gamma B, \tag{7.6}$$

$$C_m = CG^T \tag{7.7}$$

for some projective factorization (G, M, Γ) of $\hat{Q}\hat{P}$, and such that $Q, \hat{Q}, \hat{P}$ satisfy

$$0 = AQ + QA^T + D + QEQ + \tau_\perp BVB^T\tau_\perp^T, \tag{7.8}$$

$$0 = (A+QE)\hat{Q} + \hat{Q}(A+QE)^T + \hat{Q}E\hat{Q} + BVB^T - \tau_\perp BVB^T\tau_\perp^T, \tag{7.9}$$

$$0 = (A+QE)^T\hat{P} + \hat{P}(A+QE) + C^TRC - \tau_\perp^T C^TRC\tau_\perp, \tag{7.10}$$

$$\text{rank } \hat{Q} = \text{rank } \hat{P} = \text{rank } \hat{Q}\hat{P} = n_m. \tag{7.11}$$

Furthermore, the auxiliary cost is given by

$$\mathcal{J}(\mathcal{Q}, A_m, B_m, C_m) = \text{tr } QC^TRC. \tag{7.12}$$

Conversely, if $Q, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfy (7.8)–(7.11) then $(\mathcal{Q}, A_m, B_m, C_m)$ given by (7.4)–(7.7) satisfy $\mathcal{Q} \geq 0$ and (6.2) with auxiliary cost given by (7.12).

Proof: See the Appendix. □

Theorem 7.1 presents necessary conditions for the auxiliary minimization problem which explicitly characterize extremals $(\mathcal{Q}, A_m, B_m, C_m)$. These necessary conditions consist of a system of two modified Lyapunov equations and one modified Riccati equation coupled by an oblique projection τ and uncertainty terms. Setting D and E to zero, i.e., deleting the plant uncertainties, it can be seen that (7.8) drops out while (7.9) and (7.10) reduce to the optimal projection equations for model reduction obtained in [1]. If, alternatively, $n_m = n$, then the full-order robust model is given by $A + QE$, B, C where Q is given by (7.8) with $\tau_\perp = 0$.

Remark 7.2: As in the perfect modeling case considered in [1], (7.8)–(7.10) may support multiple solutions. When uncertainty is present but a full-order model is desired, then the solution is unique.

Remark 7.3: The conservatism of the bound (7.12) is difficult to predict for two reasons. First, the overbounding (4.3) holds with respect to the partial ordering of the nonnegative-definite matrices for which no scalar measure of conservatism is available. And, second, the bound (4.3) is required to hold for all nonnegative-definite matrices $\mathcal{Q}$. The conservatism will thus depend upon the actual value of $\mathcal{Q}$ determined by solving (6.2). Numerical experience with related bounds shows that the conservatism is highly problem dependent. See [8].

VIII. Sufficient Conditions for Robust Reduced-Order Modeling

The main result guaranteeing robust model reduction can now be stated.

Theorem 8.1: Suppose there exist $Q, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying (7.8)–(7.11) and suppose that $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is stabilizable for all $\Delta A \in \mathcal{U}$ with A_m, B_m, C_m given by (7.5)–(7.7) and $\mathcal{U}$ defined by (5.1). Then A_m is asymptotically stable, $A + \Delta A$ is asymptotically stable for all $\Delta A \in U$, and the model-reduction criterion satisfies the bound

$$J(A_m, B_m, C_m) \leq \text{tr } QC^TRC. \tag{8.1}$$

Proof: The converse of Theorem 7.1 implies that $\mathcal{Q}$ given by (7.4) is nonnegative definite and satisfies (6.2). With the stabilizability assumption the result follows from Proposition 6.1. □

Remark 8.1: As noted in Remark 4.1, Theorem 8.1 is effectively limited to systems which remain asymptotically stable over the class of uncertainties.

Appendix

Proof of Theorem 7.1

To optimize (6.1) over the open set

$$\mathcal{S}' \triangleq \{(\mathcal{Q}, A_m, B_m, C_m) \in \mathcal{S} : (6.3) \text{ is satisfied}\}$$

subject to the constraint (6.2), form the Lagrangian

$$\mathcal{L}(A_m, B_m, C_m, \mathcal{Q}, \mathcal{P}, \lambda) \triangleq$$

$$\text{tr}\left\{\lambda\mathcal{Q}\tilde{R} + \left[\tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \sum_{i=1}^{p} \tilde{D}_i\bar{M}_i\tilde{D}_i^T + \mathcal{Q}\tilde{E}_i^T\bar{N}_i\tilde{E}_i\mathcal{Q} + \tilde{V}\right]\mathcal{P}\right\},$$

where the Lagrange multipliers $\lambda \geq 0$ and $\mathcal{P} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ are not both zero. We thus obtain

$$\frac{\partial\mathcal{L}}{\partial\mathcal{Q}} = \tilde{A}^T\mathcal{P} + \mathcal{P}\tilde{A} + \sum_{i=1}^{p} [\tilde{E}_i^T\bar{N}_i\tilde{E}_i\mathcal{Q}\mathcal{P} + \mathcal{P}\mathcal{Q}\tilde{E}_i^T\bar{N}_i\tilde{E}_i] + \lambda\tilde{R}.$$

Setting $\partial\mathcal{L}/\partial\mathcal{Q} = 0$ yields

$$\tilde{\mathcal{A}}^T \text{ vec } \mathcal{P} = -\lambda \text{ vec } \tilde{R} \tag{A.1}$$

where "vec" is the column-stacking operation (see [6]). Since $\tilde{\mathcal{A}}$ is assumed to be stable, $\tilde{\mathcal{A}}^T$ is invertible, and thus $\lambda = 0$ implies $\mathcal{P} = 0$. Hence, it can be assumed without loss of generality that $\lambda = 1$. Furthermore, the stability of $\tilde{\mathcal{A}}^T$ implies that $\mathcal{P}$ is nonnegative definite.

Now partition $\tilde{n} \times \tilde{n}$ $\mathcal{Q}$, $\mathcal{P}$ into $n \times n$, $n \times n_m$, and $n_m \times n_m$ subblocks as

$$\mathcal{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}.$$

Thus, the stationarity conditions are given by

$$\frac{\partial\mathcal{L}}{\partial\mathcal{Q}} = \tilde{A}^T\mathcal{P} + \mathcal{P}\tilde{A} + \sum_{i=1}^{p} [\tilde{E}_i^T\bar{N}_i\tilde{E}_i\mathcal{Q}\mathcal{P} + \mathcal{P}\mathcal{Q}\tilde{E}_i^T\bar{N}_i\tilde{E}_i] + \tilde{R} = 0, \tag{A.2}$$

$$\frac{\partial\mathcal{L}}{\partial A_m} = P_{12}^T Q_{12} + P_2 Q_2 = 0, \tag{A.3}$$

$$\frac{\partial\mathcal{L}}{\partial B_m} = P_{12}^T BV + P_2 B_m V = 0, \tag{A.4}$$

$$\frac{\partial\mathcal{L}}{\partial C_m} = -RCQ_{12} + RC_m Q_2 = 0. \tag{A.5}$$

Expanding (6.2) and (A.2) yields

$$0=AQ_1+Q_1A^T+D+Q_1EQ_1+BVB^T, \quad (A.6)$$

$$0=AQ_{12}+Q_{12}A_m^T+Q_1EQ_{12}+BVB_m^T, \quad (A.7)$$

$$0=A_mQ_2+Q_2A_m^T+Q_{12}^TEQ_{12}+B_mVB_m^T, \quad (A.8)$$

$$0=A^TP_1+P_1A+E(P_1Q_1+P_{12}Q_{12})^T+(P_1Q_1+P_{12}Q_{12})E+C^TRC, \quad (A.9)$$

$$0=A^TP_{12}+P_{12}A_m+E(P_{12}^TQ_1+P_2Q_{12}^T)^T-C^TRC_m, \quad (A.10)$$

$$0=A_m^TP_2+P_2A_m+C_m^TRC_m. \quad (A.11)$$

Now define the $n \times n$ nonnegative-definite matrices

$$Q \triangleq Q_1-Q_{12}Q_2^{-1}Q_{12}^T,\ P \triangleq P_1-P_{12}P_2^{-1}P_{12}^T,$$

$$\hat{Q}=Q_{12}Q_2^{-1}Q_{12}^T,\ \hat{P}=P_{12}P_2^{-1}P_{12}^T$$

and the $n_m \times n$, $n_m \times n_m$, $n_m \times n$ matrices

$$G=Q_2^{-1}Q_{12}^T,\ M=Q_2P_2,\ \Gamma=-P_2^{-1}P_{12}^T.$$

The existence of Q_2^{-1} and P_2^{-1} follows from the fact that (A_m, B_m, C_m) is minimal. See [1]-[4], [6] for details.

Note that (A.3) implies (7.1) and (7.2). Sylvester's inequality yields (7.11). Next (7.4), (7.6), and (7.7) follow from the definition of $\mathfrak{Q}$, relations (A.4) and (A.5), and the identities

$$Q_1=Q+\hat{Q},\ Q_{12}=\hat{Q}\Gamma^T,\ P_{12}=-\hat{P}G^T,\ Q_2=\Gamma\hat{Q}\Gamma^T,\ P_2=G\hat{P}G^T.$$

Computing either Γ(A.7), (A.8) or G(A.10) + (A.11) yields (7.5). Inserting (7.5)-(7.7) into (A.6)-(A.11) it can be shown that (A.8), (A.9), and (A.11) are superfluous. Using (A.6) + $G^T\Gamma$(A.7)G − (A.7)G − $[(A.7)G]^T$ and $G^T\Gamma$(A.7)G − (A.7)G − $[(A.7)G]^T$ yield (7.8) and (7.9). Similarly, Γ^TG(A.10)Γ − (A.10)Γ − $[(A.10)\Gamma]^T$ yields (7.10).

Finally, the proof can be reversed so that (7.5)-(7.11) yield (A.1)-(A.5) and (6.2).

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Defect Correction Methods for the Solution of Algebraic Riccati Equations

V. MEHRMANN AND E. TAN

Abstract—**The solution of discrete and continuous algebraic Riccati equations is considered. It is shown that if an approximate solution is obtained, then the defect for this solution again solves an algebraic Riccati equation of the same form and that the system properties of detectability and stabilizability are inherited by this defect equation. On the basis of these results, a general defect correction method is proposed and numerical examples are given for the use of this method in combination with the SR method.**

I. Introduction

We consider the numerical solution of generalized algebraic Riccati equations

$$0=A^*XE+E^*XA-(B^*XE+S^*)^*R^{-1}(B^*XE+S^*)+C^*QC=\mathfrak{CR}(X) \quad (1.1)$$

$$0=-E^*XE+A^*XA-(A^*XB+S)(R+B^*XB)^{-1}\cdot(A^*XB+S)^*+C^*QC=\mathfrak{DR}(X) \quad (1.2)$$

where $X, A, E \in \mathbb{C}^{n,n}$, $B, S \in \mathbb{C}^{n,m}$, $C \in \mathbb{C}^{r,n}$, $Q = Q^* \in \mathbb{C}^{r,r}$, $R = R^* \in \mathbb{C}^{m,m}$ are positive definite and E is nonsingular. Both equations arise, for example, in the solutions of linear quadratic optimal control problems. Equation (1.1) stems from a continuous-time and (1.2) from a discrete-time problem (see, e.g., [1], [4], and [5]). The numerical solution of these two equations has been studied extensively in recent years (e.g., [7], [4]-[6], [9], [14], [16], [17], and [20]). Typically, solutions are obtained using QR- or QZ-type algorithms to compute deflating subspaces of the matrix pencils

$$\begin{bmatrix} A & 0 & B \\ C^*QC & A^* & S \\ S^* & B^* & R \end{bmatrix} - \lambda \begin{bmatrix} E & 0 & 0 \\ 0 & -E^* & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (1.3)$$

corresponding to problem (1.1) and

$$\begin{bmatrix} A & 0 & B \\ C^*QC & -E^* & S \\ S^* & 0 & R \end{bmatrix} - \lambda \begin{bmatrix} E & 0 & 0 \\ 0 & -A^* & 0 \\ 0 & -B^* & 0 \end{bmatrix} \quad (1.4)$$

corresponding to problem (1.2), where, using the fact that R is positive definite, many of the algorithms are applied to the reduced pencils

$$\begin{bmatrix} F & G \\ H & -F^* \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \\ 0 & E^* \end{bmatrix} = \begin{bmatrix} A-BR^{-1}S^* & BR^{-1}B^* \\ C^*QC-SR^{-1}S^* & -A^*+SR^{-1}B^* \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \\ 0 & E^* \end{bmatrix} \quad (1.5)$$

corresponding to (1.3) and

$$\begin{bmatrix} F & 0 \\ H & E^* \end{bmatrix} - \lambda \begin{bmatrix} E & -G \\ 0 & F^* \end{bmatrix} = \begin{bmatrix} A-BR^{-1}S & 0 \\ C^*QC-SR^{-1}S^* & E^* \end{bmatrix} - \lambda \begin{bmatrix} E & -BR^{-1}B^* \\ 0 & A^*-SR^{-1}B^* \end{bmatrix} \quad (1.6)$$

corresponding to (1.4).

Manuscript received February 13, 1987, revised September 28, 1987. Paper recommended by Associate Editor, B. H. Krogh.

The authors are with the Fakultät für Mathematik, Universität Bielefeld, Bielefeld, West Germany.

IEEE Log Number 8821013.

0018-9286/88/0700-0695$01.00 © 1988 IEEE