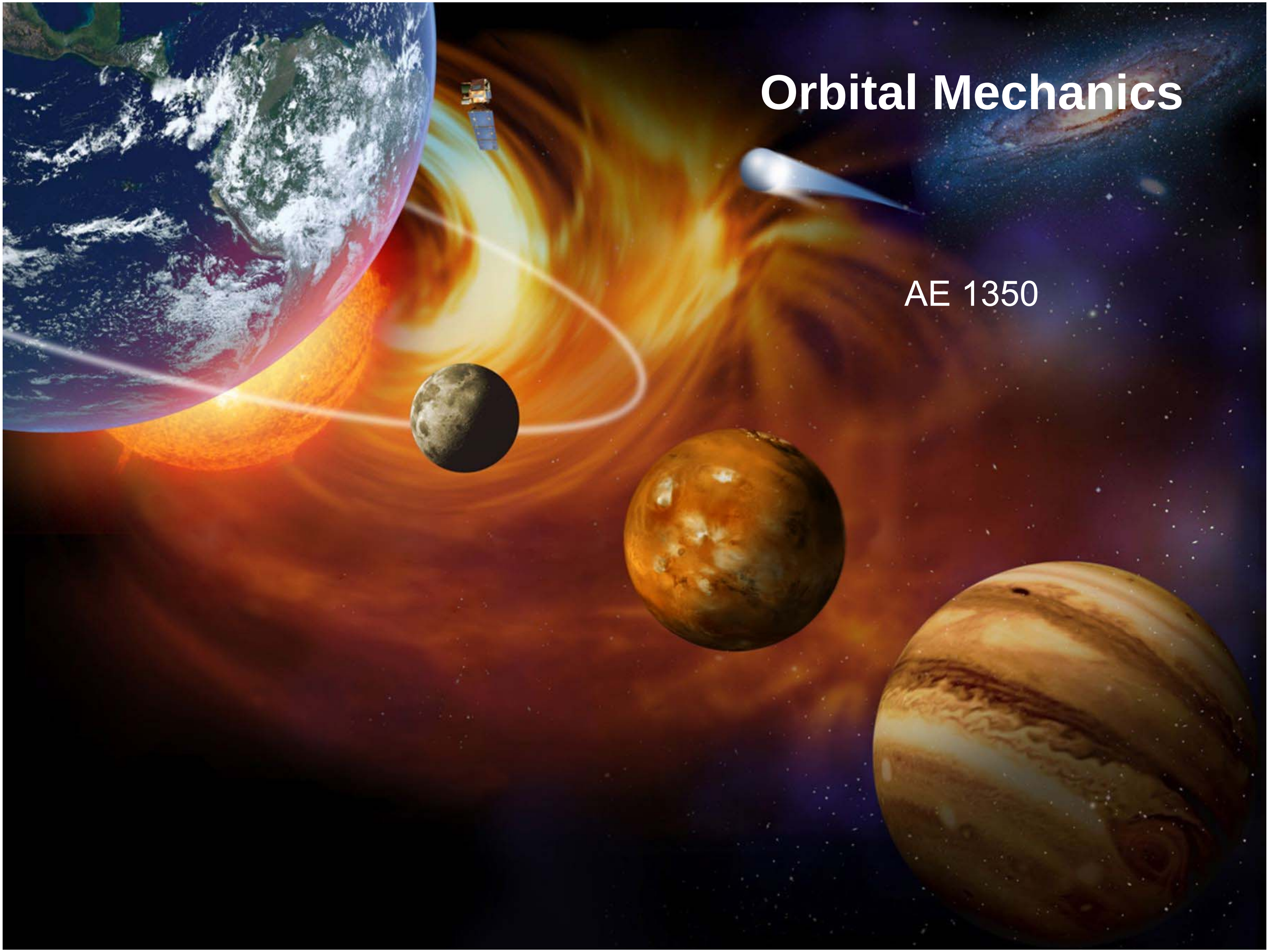

Orbital Mechanics
AE 1350

Outline

- Newton's law of gravitation
- The N-body problem
- Two-body motion
- Conic sections
- Orbit properties
- Orbital elements
- A few examples

*Read Chapter 8 in your text

Newton's Universal Law of Gravitation and the N-Body Problem

Newton's Law of Universal Gravitation

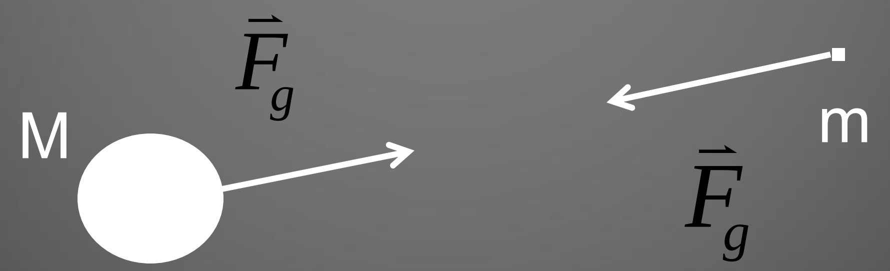
- Besides stating his 3 laws of motion in the Principia, Newton presented a law of universal gravitation, stating that:
  - Any two bodies attract one another with a force proportional to the product of their masses and inversely proportional to the square of the distance between them

Newton's Law of Universal Gravitation (cont.)

$$\boxed{\vec{F}_g = \frac{-GMm}{r^2}\hat{r}} \qquad (1)$$

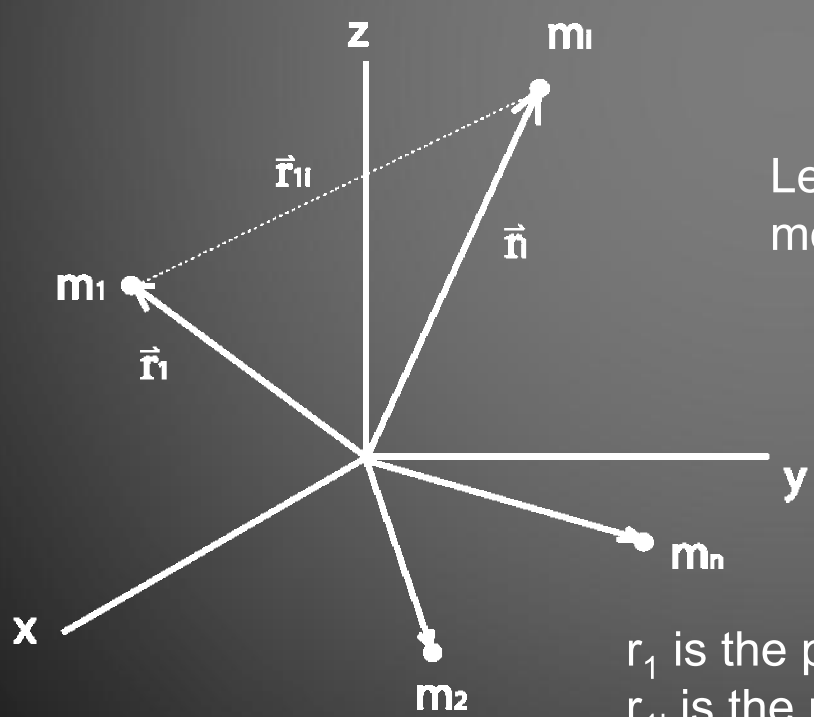
where G = 6.674 x 10^{-11} Nm^2/kg^2

This assumes a <u>spherical mass distribution</u> of the gravitational bodies.



The N-Body Problem

- At any given time, a body in space is being acted upon by several gravitational forces, and other forces like drag, solar pressure, thrust, etc.
- If we consider just the gravitational forces, assuming each body has a spherical mass distribution:



Let's study the motion of body m_i

r_1 is the position of the body 1 relative to our origin
r_{1i} is the position of the body i relative to body 1

The N-Body Problem (cont.)

Applying Newton's Law of Universal Gravitation between body m_i & m_1 :

$$\vec{F}_{g_1} = \frac{-Gm_i m_1}{r_{1i}^{\ 2}} \hat{r}_{1i}$$

where: $\vec{r}_{1i} = \vec{r}_i - \vec{r}_1$

r_1 is the position of the body 1 relative to our origin
r_{1i} is the position of the body i relative to body 1
F_{g1} is the gravitational force exerted on m_i by m_1

The N-Body Problem (cont.)

The vector sum of all such gravitational forces action on body m_i is:

$$\vec{F}_g = \sum_{\substack{j=1 \\ j \neq i}}^{n} \vec{F}_{g_j} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{-Gm_i m_j}{{r_{ji}}^2} \hat{r}_{ji}$$

$$\boxed{\vec{F}_g = -Gm_i \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j}{{r_{ji}}^2} \hat{r}_{ji}} \qquad (2)$$

N-Body Equation of Motion

From Newton's 2nd Law, applied in an inertial frame:

$$\frac{d}{dt}\left(m_i \vec{v}_i\right) = \vec{F}$$

$$m_i \frac{d\vec{v}_i}{dt} + \vec{v}_i \frac{dm_i}{dt} = \vec{F}$$

Noting that:

$$\vec{v}_i = \dot{\vec{r}}_i$$

$$\frac{d\vec{v}_i}{dt} = \ddot{\vec{r}}_i$$

We have:

$$m_i \ddot{\vec{r}}_i + \dot{\vec{r}}_i \dot{m}_i = \vec{F}$$

$$\boxed{\ddot{\vec{r}}_i = \left(\frac{\vec{F}}{m_i}\right) - \dot{\vec{r}}_i \left(\frac{\dot{m}_i}{m_i}\right)} \quad (3)$$

Where $\ddot{\vec{r}}_i$ is defined as the acceleration of the i^{th} body relative to our xyz coordinate system, and:

$$\vec{F} = \vec{F}_g + \vec{F}_{other}$$

$$\vec{F}_{other} = \vec{F}_{drag} + \vec{F}_{thrust} + \vec{F}_{\substack{solar\\pressure}} + \ldots$$

N-Body Equation of Motion (cont.)

Equation (3) is a 2nd order, nonlinear vector differential equation that has defied closed-form solution, primarily because of the form of F and presence of m_i dot term. This equation can be, and is generally, solved via numerical integration.

However if we assume: (1) m_i = constant, m_i dot = 0, and
(2) $F_{other} = 0$

Then we are back to an equation like (2) in which the equation of motion is dominated by gravitational forces:

$$\boxed{\ddot{\vec{r}}_i = \frac{\vec{F}_g}{m_i} = -G\sum_{\substack{j=1 \\ i\neq j}}^{n} \frac{m_j}{r_{ji}^2}\hat{r}_{ji}} \quad (4)$$

Georgia Tech

An Example

Body 1: Earth

Body 2: S/C

Body 3: Moon

Body 4: Sun

$$\ddot{\vec{r}}_1 = -G\sum_{j=2}^{n}\frac{m_j}{r_{j1}^2}\hat{r}_{j1}$$

$$\ddot{\vec{r}}_2 = -G\sum_{\substack{j=1\\ j\neq 2}}^{n}\frac{m_j}{r_{j2}^2}\hat{r}_{j2}$$

As we defined earlier:

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$$

$$\ddot{\vec{r}}_{12} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1$$

where $\vec{r}_{12}$ is the position of body 2 relative to body 1 and $\ddot{\vec{r}}_{12}$ is the acceleration of body 2 relative to body 1

$$\ddot{\vec{r}}_{12} = -G\left[\frac{m_1}{r_{12}^2}\hat{r}_{12} + \sum_{j=3}^{n}\frac{m_j}{r_{j2}^2}\hat{r}_{j2} - \frac{m_2}{r_{21}^2}\overset{-\hat{r}_{12}}{\hat{r}_{21}} - \sum_{j=3}^{n}\frac{m_j}{r_{j1}^2}\hat{r}_{j1}\right]$$

$$\boxed{\ddot{\vec{r}}_{12} = \frac{-G(m_1+m_2)}{r_{12}^2}\hat{r}_{12} - G\sum_{j=3}^{n} m_j\left(\frac{\hat{r}_{j2}}{r_{j2}^2} - \frac{\hat{r}_{j1}}{r_{j1}^2}\right)} \quad (5)$$

Components of Equation (5)

Equation (5) can be used to describe the acceleration (motion) of the spacecraft relative to the Earth, the 2nd term on the RHS of Eq. (5) accounts for the "perturbing effects" of other bodies

$$\ddot{\vec{r}}_{12} = \underbrace{\frac{-G(m_1+m_2)}{r_{12}^{\;2}}\hat{r}_{12}}_{(1)} \underbrace{- G\sum_{j=3}^{n} m_j\left(\underbrace{\frac{\hat{r}_{j2}}{r_{j2}^{\;2}}}_{(3)} - \underbrace{\frac{\hat{r}_{j1}}{r_{j1}^{\;2}}}_{(4)}\right)}_{(2)}$$

$\ddot{\vec{r}}_{12}$ Acceleration of body 2 relative to body 1

(1) two body term which is a function of m_1, m_2, r_{12}
(2) gravitational acceleration caused by masses 3->n. "Perturbing Term"
(3) gravitational acceleration between masses 3->n and body 2. Acceleration of body 2 relative to body j
(4) gravitational acceleration between masses 3->n and body 1. Acceleration of body 1 relative to body j.

Relative Acceleration for a Low Earth Orbit Satellite

Relative acceleration in g's for a 200-NM Earth satellite:

Earth	0.89 (Spherical)
Earth oblateness	10^{-3}
Sun	6×10^{-4}
Moon	3×10^{-6}
Jupiter	3×10^{-8}
Venus	2×10^{-8}
Saturn	2×10^{-9}
Mars	7×10^{-10}

Solar radiation pressure provide a perturbation of about this same magnitude

In this table, relative acceleration is defined as:

$$\frac{-Gm_j}{r_j^2}\hat{r}_j$$

which is equivalent to the:

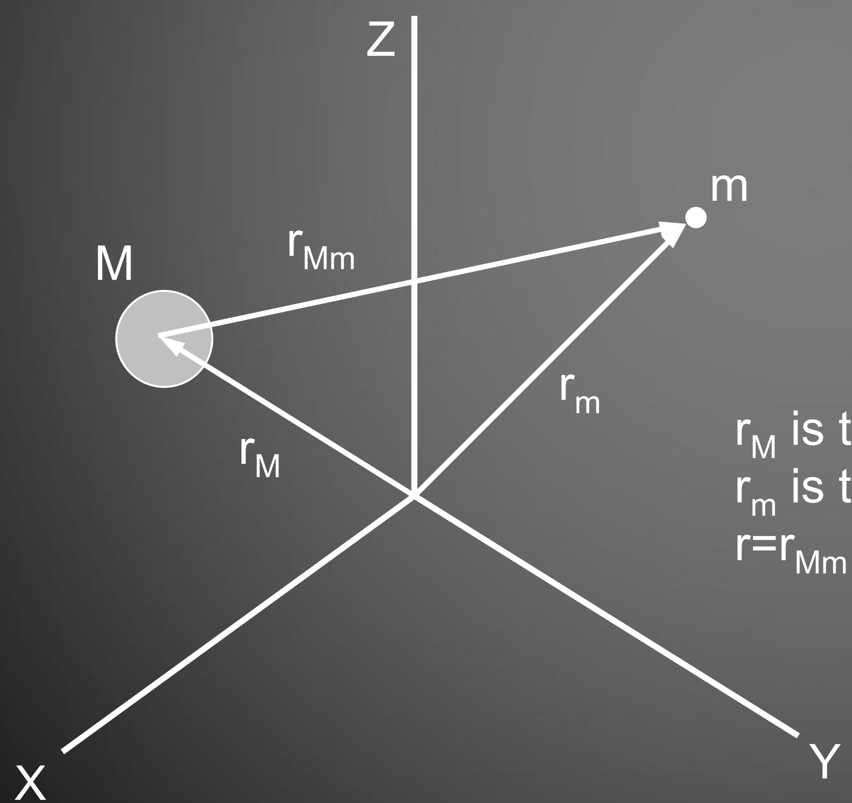
$$\frac{-Gm_j\hat{r}_{j_2}}{r_{j_2}^2}$$

term in Eq. (5)

Two Body Motion

The Two Body Problem

1) Spherical mass distribution of all gravitational bodies
2) No forces acting on system other than gravity
3) No mass change



r_M is the position of body M relative to the origin
r_m is the position of body m relative to the origin
$r=r_{Mm}$ as position of body m relative to body M

Two Body Equation of Relative Motion

Applying Newton's 2nd law in an inertia reference frame to body m yields:

$$\sum \vec{F} = \vec{F}_g = m\ddot{\vec{r}}_m$$

where from Newton's law of universal gravitation (1):

$$-G\frac{Mm}{r^2}\hat{r} = m\ddot{\vec{r}}_m$$

$$\ddot{\vec{r}}_m = \frac{-GM\hat{r}}{r^2}$$

Two-Body Equation of Relative Motion (cont.)

Similarly for body M:

$$\ddot{\vec{r}}_M = \frac{Gm\hat{r}}{r^2}$$

And since:

$$\vec{r} = \vec{r}_m - \vec{r}_M$$
$$\ddot{\vec{r}} = \ddot{\vec{r}}_m - \ddot{\vec{r}}_M$$

$$\boxed{\ddot{\vec{r}} = -\frac{G(M+m)\hat{r}}{r^2}} \quad (6)$$

Two-Body Equation of Relative Motion (cont.)

Unlike equations (3) and (5), this 2nd order vector differential equation has a closed form solution. As one would expect, note that Eq. (6) is the same as that developed for N-body motion (Eq. (5)) without the other body perturbing effects.

For most 2-body problems, we have a spacecraft orbiting a much more massive planetary body: m<<M

$$G(m+M) \approx GM = \mu \qquad \mu \text{ is in units of } \left(\frac{km^3}{s^2}\right)$$

$$\boxed{\ddot{\vec{r}} = \frac{-\mu}{r^2}\hat{r}} \qquad (7)$$

Energy

Kinetic energy = $1/2mv^2$

Potential energy = -W where:

$$\Delta PE = -W = -\int_{x_i}^{x_f} F(x)\,dx$$

Work done equals the decrease in PE

$$\Delta PE = -\int_{ref}^{r} \vec{F}_g\,dr = -\int_{ref}^{r} \frac{-GMm}{r^2}\hat{r}\,dr$$

$$\Delta PE = \left.\frac{-GMm}{r}\right|_{ref}^{r}$$

$$PE\big|_r - PE\big|_{ref} = -\frac{-GMm}{r} + \frac{GMm}{r_{ref}}$$

Total (Specific) Energy

Define $PE\big|_{ref} = 0$ @ $r_{ref} = \infty$ therefore,

$$PE = -\frac{GMm}{r} = \frac{-\mu m}{r}$$ (PE will always be negative)

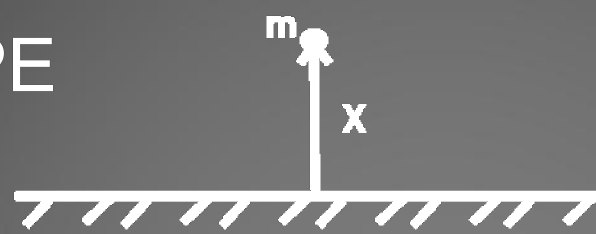
Total Energy per unit mass:

$$\varepsilon = \frac{KE + PE}{m} = \frac{v^2}{2} - \frac{\mu}{r}$$

$$\boxed{\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}} \quad (8)$$

Comparison of PE Relations

High School PE



$$\boxed{PE = mg_0 h}$$

Note from :

$$\vec{F}_g = -mg = -\frac{GMm}{r^2}\hat{r}$$

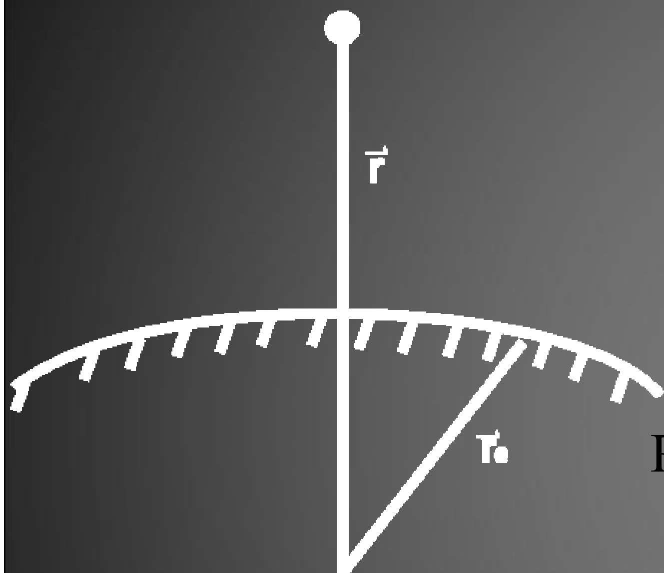
where g = g(r)

At surface,
$g=g_0$ and
$r=r_e$, so:

$$g_0 = \frac{\mu}{r_e^{\,2}}$$

$$PE = \left(\frac{mh\mu}{r_e^{\,2}}\right)$$

Orbital PE



If we change the datum to $r_{ref} = r_e$, then

$$PE\big|_{re} = 0$$

$$PE\big|_{r} = -\frac{GMm}{r} + \frac{GMm}{r_e}$$

$$PE = -\frac{GMm}{(r_e + h)} + \frac{GMm}{r_e} = GMm\left[\left(\frac{-r_e}{r_e + h}\right)\left(\frac{1}{r_e}\right) + \frac{(r_e + h)}{r_e(r_e + h)}\right]$$

$$PE = \frac{\mu m h}{r_e^{\,2}\left(1 + \frac{h}{r_e}\right)}$$

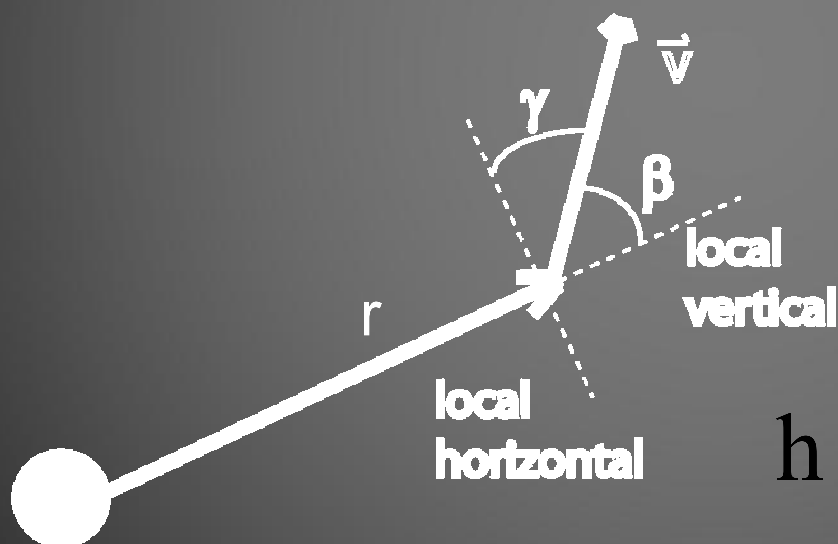
So, as (h/r_e) -> 0 or $h << r_e$ the PE = mg_0h approximation is valid. However, for orbital mechanics, where h may be greater than r_e

$$\boxed{PE = -\frac{\mu m}{r}}$$

Specific Angular Momentum

Angular Momentum: $\boxed{\vec{h} = \vec{r} \times \vec{v}}$ (9)

Since h is the cross-product of r and v, it must always be perpendicular to the plane containing r and v



γ: flight path angle
β: zenith angle

$$h = rv\sin\beta = rv\cos\gamma$$

Constants of Two-Body Motion

- Two body motion is conservative.
- An object moving under the influence of a single gravitational field, defined with spherical, constant mass distribution
  – Does not lose or gain energy, but simply exchanges PE for KE and vice versa (constant ε)
  – Has constant angular momentum (h), both magnitude and direction

Two-Body Trajectory Equation

$$\ddot{\vec{r}} = -\frac{\mu}{r^2}\hat{r}$$

One can analytically solve the two-body equation of motion (7), yielding:

$$\boxed{r = \frac{h^2/\mu}{\left(1 + \frac{B}{\mu}\cos\theta\right)}} \qquad (10)$$

Equation 10 is the 2-body trajectory equation, written in polar coordinates where r is given as a function of θ and the constants of motion (h, μ, B). Here, θ is defined as the angle between the B and r vectors. B is a vector constant of integration that has yet to be defined.

Conic Sections and Their Implications for the Two-Body Problem

Kepler's Laws

- First Law: The orbit of each planet is an ellipse, with the Sun at a focus (1609)
- Second Law: The line joining the planet to the Sun sweeps out equal areas in equal times (1609)
- Third Law: The square of the period of a planet is proportional to the cube of its mean distance from the Sun (1619).

*Combining the two-body trajectory equation and the geometric properties of conic sections, will allow us to prove Kepler's laws

General Equation of a Conic Section

The general equation of a conic section, written in polar coordinates with the origin located at one focus and the angle θ defined as the angle between r and the point on the conic section nearest the focus (origin) may be expressed as:

$$\boxed{r = \frac{p}{1 + e\cos\theta}} \qquad (11)$$

Comparing Equations (10) and (11) we see that these two relations are mathematically identical if we define:

$$\boxed{p = h^2/\mu} \qquad (12)$$ p is termed the semi-latus rectum

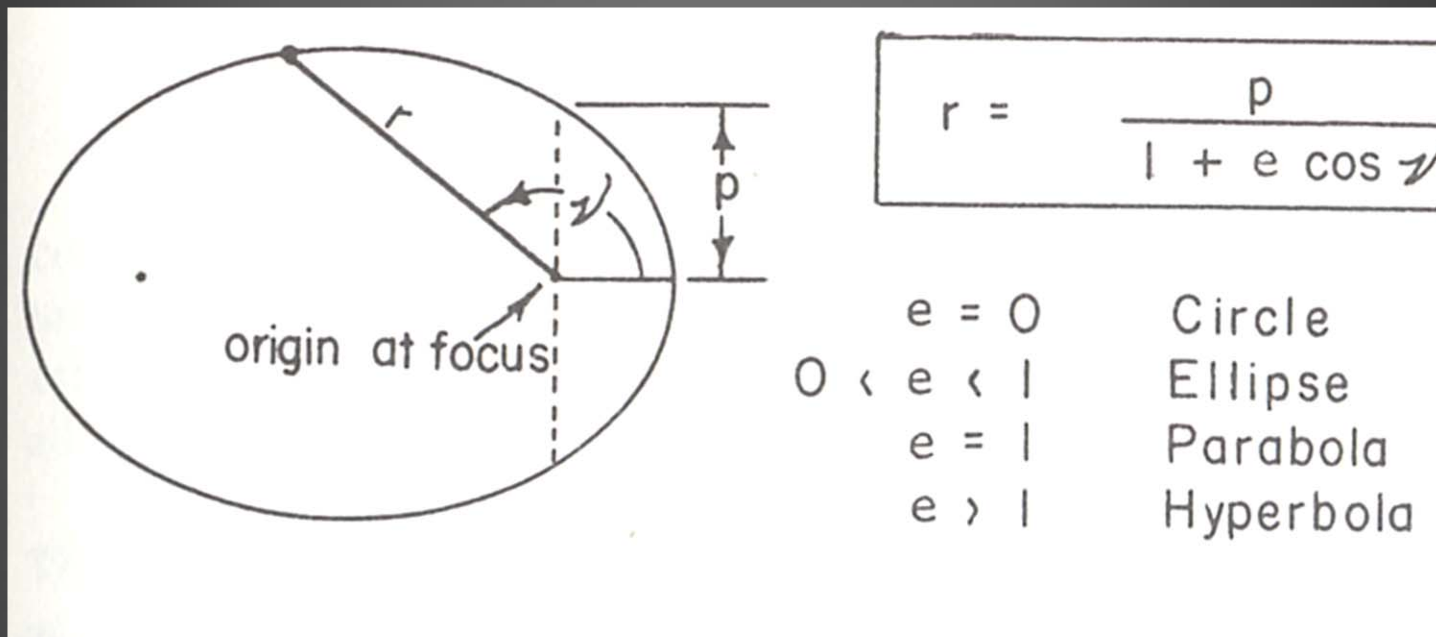
$$\boxed{e = B/\mu} \qquad (13)$$ e is termed the eccentricity

Where for a conic section,

p is a geometric constant

e determines the “type” of conic.

Semi-Latus Rectum and Eccentricity



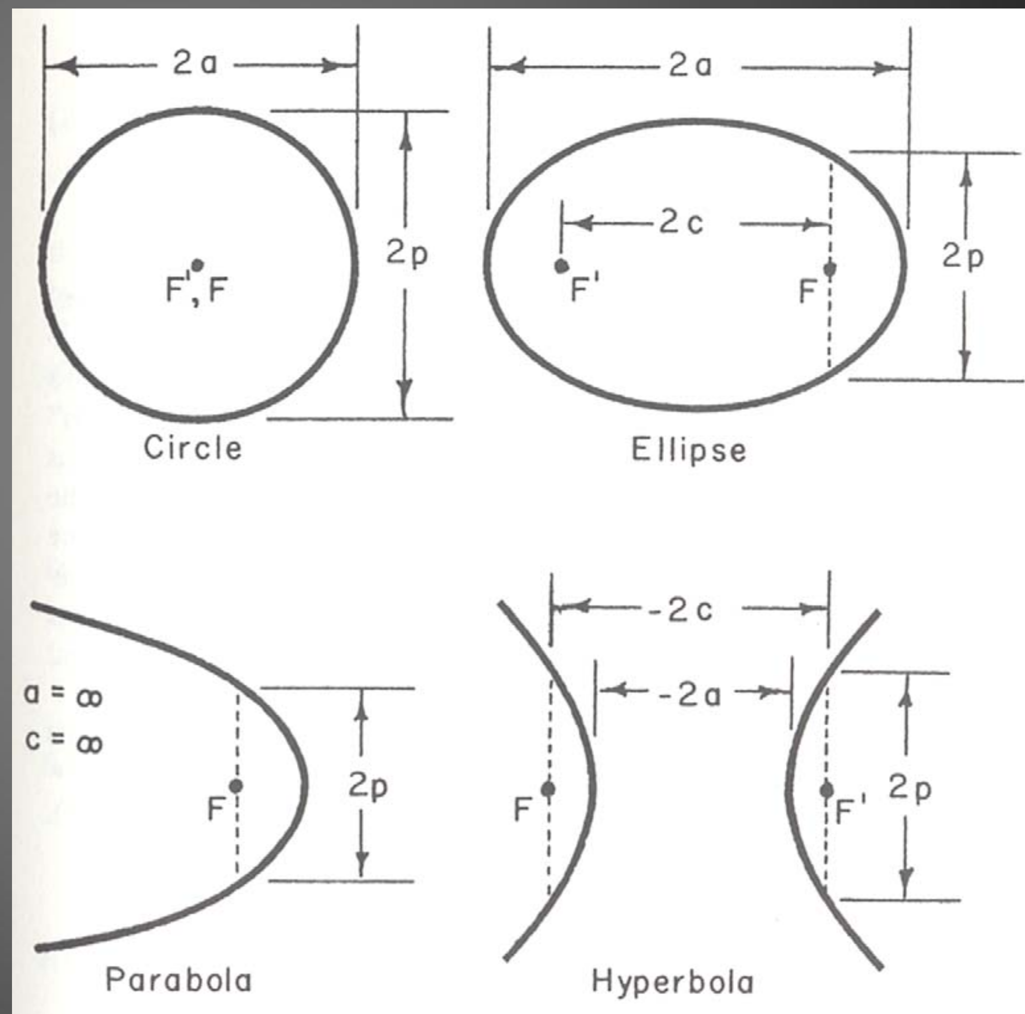
The similarity in form between the 2-body trajectory equation (10) and the general equation of a conic section (11) verifies Kepler's 1st law (for ellipses) and allows us to extend this law to include orbital motion along any general conic section.

Summary of Two-Body Motion

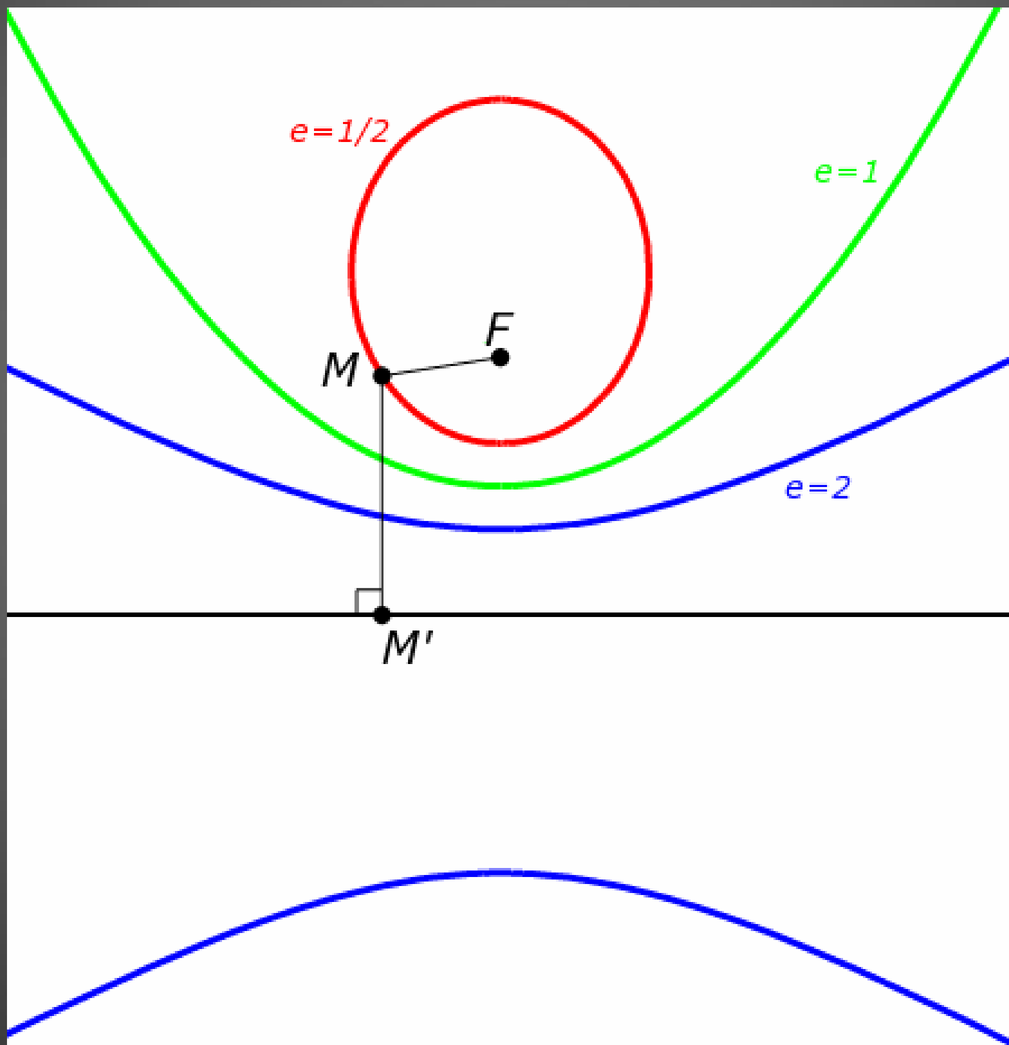
1) The family of curves called conic sections represents the <u>only</u> possible paths for an orbiting object in the 2 body problem
2) The focus of the conic orbit must be located at the center of the dominant gravitational mass.
3) E = constant. PE < 0 and PE -> 0 as r -> ∞ so the KE (and s/c speed) decreases as the s/c gains altitude and increases as the s/c loses altitude.
4) h vector is a constant, so motion takes place in a single plane, the orbital plane, fixed in inertial space. As r and v change along the orbital path, γ must change such that rvcosγ remains constant.

Conic Sections

- Known and studied for centuries, dating back to the time of the Greeks
- Conic is the locus of points defined such that the ratio of its distance from a given point (a focus) to the distance from a given line (a directix) is a positive constant, e.



Conic Sections (cont.)



Conic Sections (cont.)

Because of symmetry, all conic sections have two foci, F and F'. The prime focus, F, marks the location of the central gravitational body.

2p is the width of each curve at the focus (latus rectum)

2a is the length of the chord passing through the two focii (major axis)

2c is the distance between the two foci

where, $\boxed{e = c/a}$ (14)

and $\boxed{p = a\left(1 - e^2\right)}$ (15)

for all conic sections except parabolas

Apses

The extreme points along a conic's major axis are referred to as "apses"

periapsis, $\theta = 0^o$ (closest point), r_p

apoapsis, $\theta = 180^o$ (farthest point), r_a

So, from (11):

$$\boxed{\begin{aligned} r_p &= \frac{p}{1+e} = a(1-e) \\ r_a &= \frac{p}{1-e} = a(1+e) \end{aligned}} \qquad (16)$$

Angular Momentum

Since h = rvcosγ

And at r_p and r_a γ = 0 (from geometry)

So,

$$\boxed{h = r_p v_p = r_a v_a} \quad (17)$$

At periapsis, the energy can be calculated as:

$$E = \frac{{v_p}^2}{2} - \frac{\mu}{r_p} \qquad \text{where: } h^2 = p\mu = a\mu\left(1 - e^2\right)$$

$$E = \frac{h^2}{2{r_p}^2} - \frac{2r_p\mu}{2{r_p}^2} = \frac{h^2 - 2r_p\mu}{2{r_p}^2}$$

$$E = \frac{h^2 - 2\mu\left(a[1-e]\right)}{2\left(a[1-e]\right)^2}$$

Energy

So:

$$E = \frac{a\mu(1+e)(1-e) - 2a\mu(1-e)}{2a^2(1-e)^2}$$

$$E = \frac{\mu(1+e) - 2\mu}{2a(1-e)}$$

$$E = \frac{\mu e - \mu}{2a(1-e)}$$

$$E = \frac{\mu(e-1)}{2a(1-e)}$$

$$\boxed{E = -\frac{\mu}{2a}} \quad (18)$$

Valid for all conic sections

Circle	a>0, E<0
Ellipse	a>0, E<0
Parabola	A=∞, E=0
Hyperbola	A<0, E>0

Eccentricity

Now, since h determines p, and E determines a, the two together determine e (from Eq. 15)

$$p = a\left(1 - e^2\right)$$

$$\frac{h^2}{\mu} = \frac{-\mu}{2E}\left(1 - e^2\right)$$

$$\frac{2Eh^2}{\mu^2} = e^2 - 1$$

$$\frac{2Eh^2}{\mu^2} = e^2 - 1$$

$$\boxed{e = \left[1 + \frac{2Eh^2}{\mu^2}\right]^{1/2}} \quad (19)$$

Summary of Conic Section Parameters

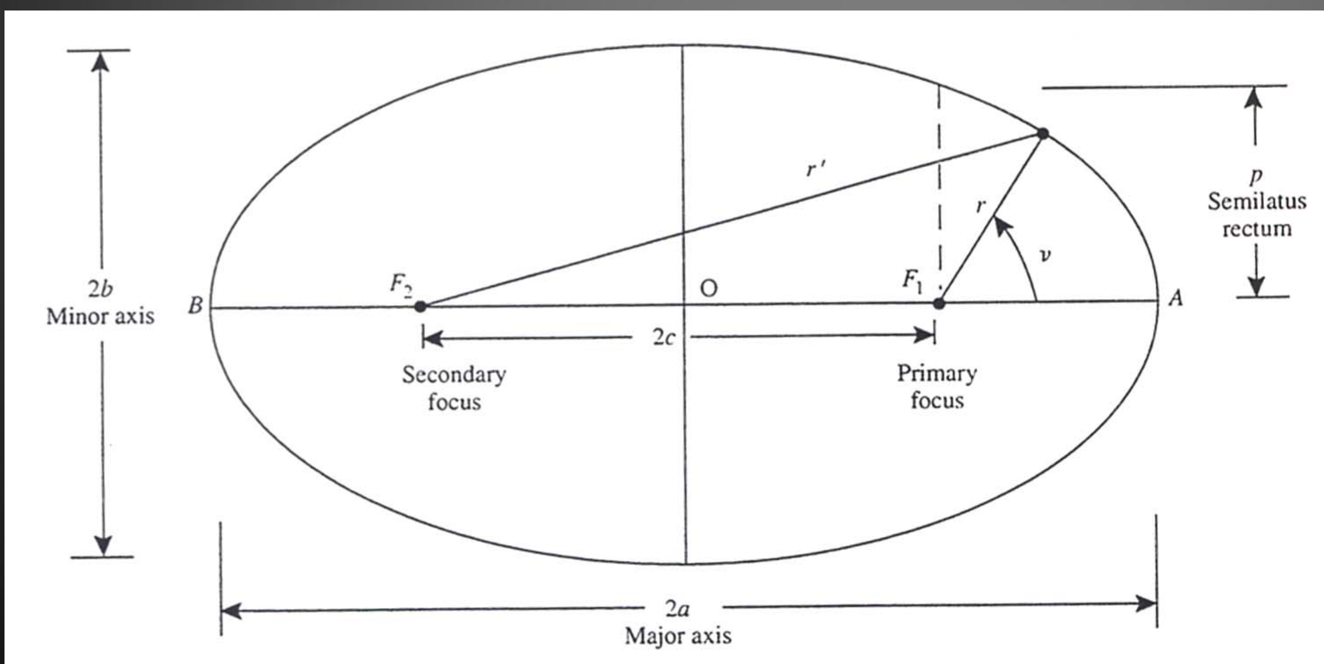
	e	a	E	
Circle	0	>0	<0	Dominated by PE
Ellipse	0<e<1	>0	<0	
Parabola	1	∞	=0	PE = KE $V_\infty = 0$
Hyperbola	>1	<0	>0	KE>PE $V_\infty > 0$

Elliptical and Circular Orbits

Properties of Elliptical Orbits

The orbits of all the planets in our solar system as well as most spacecraft are ellipses.

An ellipse can be defined geometrically as the locus of points transcribed by a piece of string anchored at the two foci and traced out for all θ



For $\theta = 0^o$ or 180^o,
$r + r' = 2a$

By definition or (16)
$r_p + r_a = 2a$

Properties (cont.)

Also by inspection, or from Eqs. (16) and (14)

$$r_a - r_p = 2c$$

And from (14),

$$e = \frac{c}{a}$$

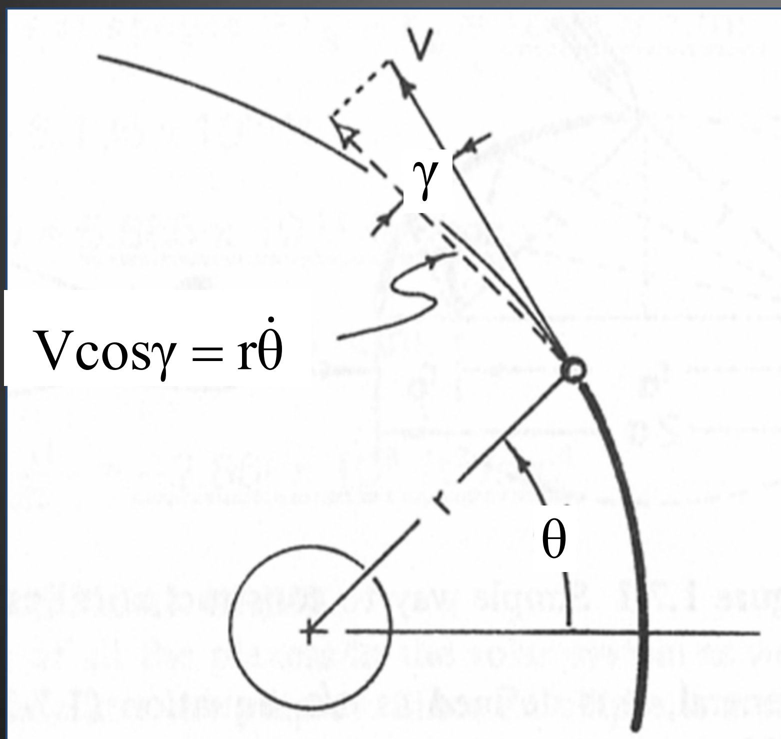
$$\boxed{e = \left(\frac{r_a - r_p}{r_a + r_p}\right)} \quad (20)$$

2b is the width of the ellipse at its center (minor axis), where by geometry:

$$\boxed{a^2 = b^2 + c^2} \quad (21)$$

Orbital Period

Orbital Period is defined as the time required for the spacecraft to travel once around its orbit (P)



tangential velocity component

$= V\cos\gamma = r\dot{\theta}$

Recall, $h = rv\cos\gamma$

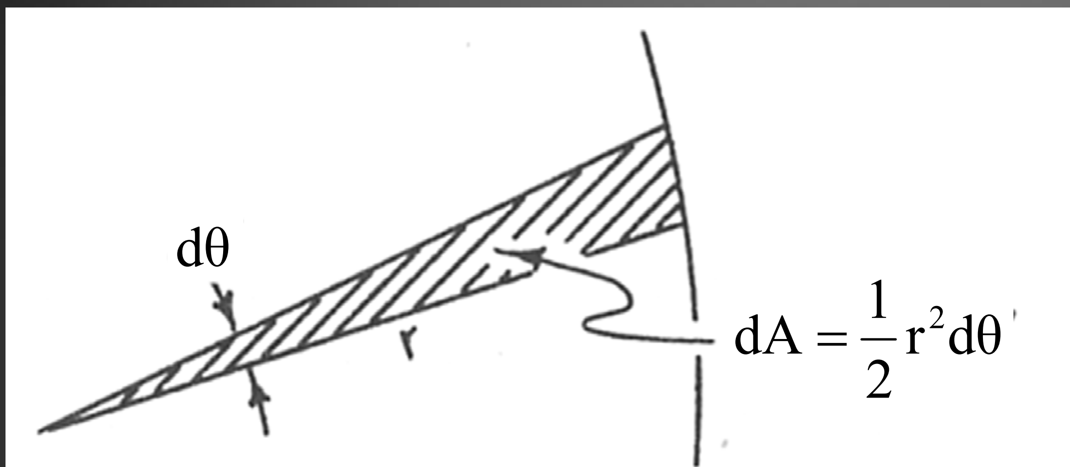
$$h = r^2\dot{\theta}$$

$$h = r^2 \frac{d\theta}{dt}$$

$$dt = \frac{r^2}{h} d\theta$$

So: $d\theta = \frac{h}{r^2} dt$

Orbital Period (cont.)



From geometry, we see that:

$$d\theta = \frac{2}{r^2} dA$$

So,

$$\boxed{dt = \frac{2}{h} dA} \qquad (22)$$

Equation (22) is a mathematical statement of Kepler's 2nd law (since h = constant). Over the course of one orbit,

$$\int_0^p dt = \frac{2}{h}\int dA = \frac{2}{h}\pi ab$$

$$P = \frac{2ab\pi}{h}$$

Orbital Period (cont.)

From Equations (21) and (14):

$$b^2 = \left(a^2 - c^2\right) = \left(a^2 - e^2 a^2\right) = a^2\left(1 - e^2\right)$$

And from Equation (15):

$$b^2 = ap \Rightarrow b = \sqrt{ap}$$

From Equation (12):

$$h = \sqrt{p\mu}$$

So,

$$P = \frac{2a\pi\sqrt{ap}}{\sqrt{p\mu}}$$

$$\boxed{P = \frac{2a^{3/2}\pi}{\sqrt{\mu}}} \qquad (23)$$

Equation 23 is a mathematical statement of Kepler's 3rd law, proving that the period of an elliptical orbit depends only on the size of the semi-major axis (a is the average of r_p and r_a, that is the mean distance)

Circular Orbits

A circular orbit is simply a special case of an elliptical orbit where e = 0 and r = constant = p = a. In this case, Equation (23) becomes,

$$P = \frac{2r^{3/2}\pi}{\sqrt{\mu}}$$

since r = a

From the energy equation, we see that:

$$E = \frac{v^2}{2} - \frac{\mu}{r} = \frac{-\mu}{2a}$$

For a circular orbit where r = a, this becomes:

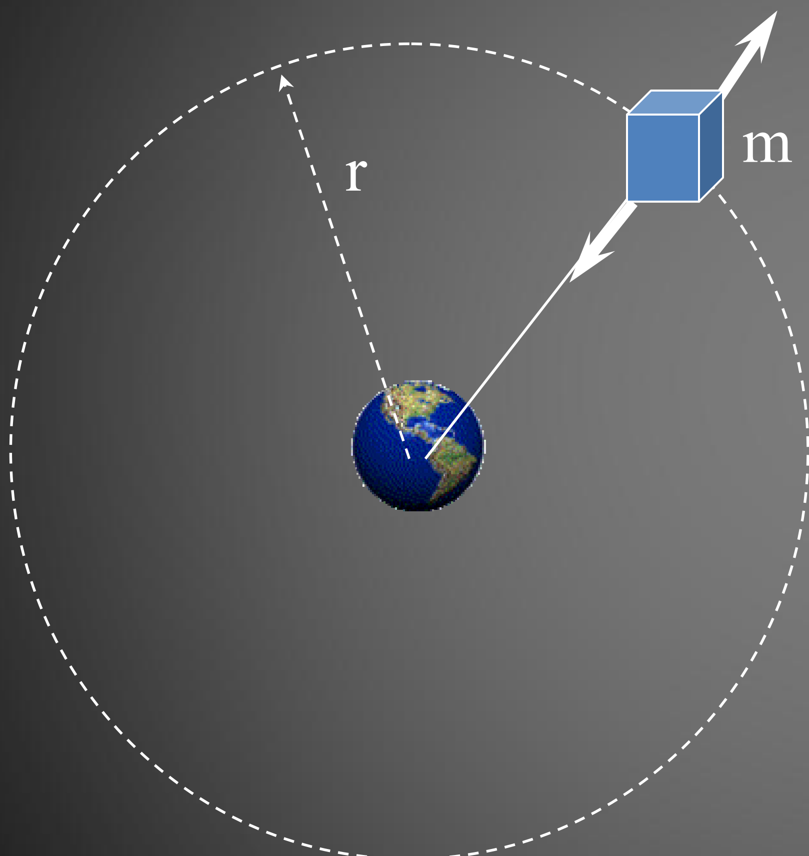
$$\frac{v^2}{2} = \frac{-\mu}{2r} + \frac{2\mu}{2r} = \frac{\mu}{2r}$$

and

$$\boxed{v_c = \left(\frac{\mu}{r}\right)^{1/2}} \qquad (24)$$

Note that v_c decreases as r increases and v_c increases as r decreases

Satellites in Circular Orbit



Centrifugal force = mV^2/r

Force due to gravity = GMm/r^2

$$GMm/r^2 = mV^2/r$$
$$GM/r = V^2$$

Here,

G = Universal constant

M = Mass of the earth

V = Velocity of the satellite

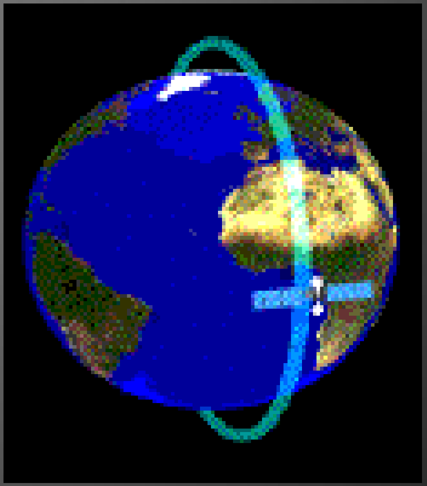
r = Radius of the orbit

Low-inclination LEO

- Inclination of orbit plane: for due east launch, inclination will be equal the launch latitude (minimum required-energy orbit)
- For example, inclination of many US manned missions is approximately the latitude of NASA KSC, about 28.5 degrees
- The lower the latitude, the larger the amount of “free” velocity provided by the rotation of the Earth
  - Cheaper to launch from Florida than Virginia…

Polar LEO

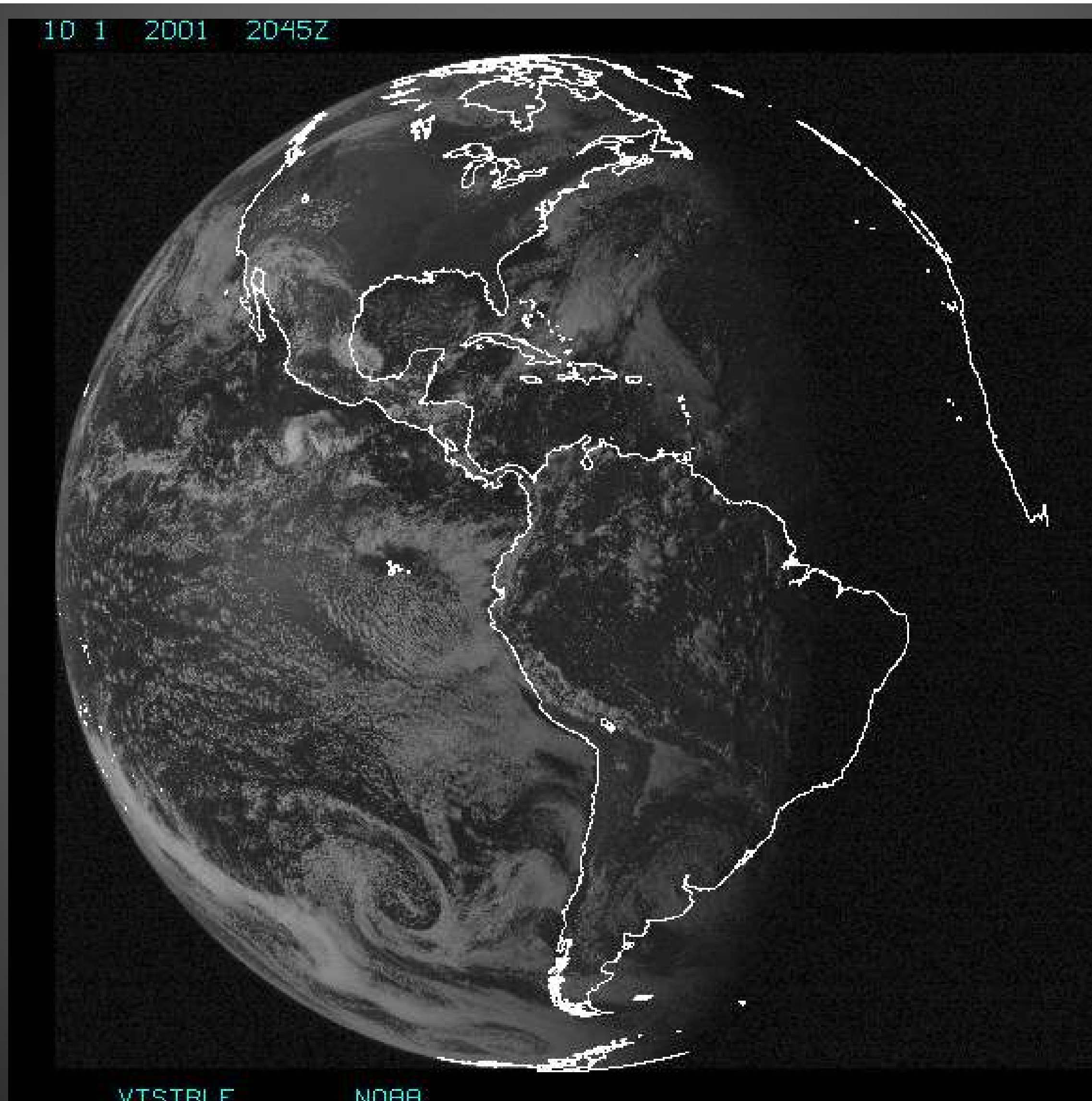
- Polar orbits have an inclination near 90 deg
- Have the advantage that they pass over the entire planet at regular intervals
- Takes more energy to get there than for low inclination orbits



Geostationary Orbit (GEO)

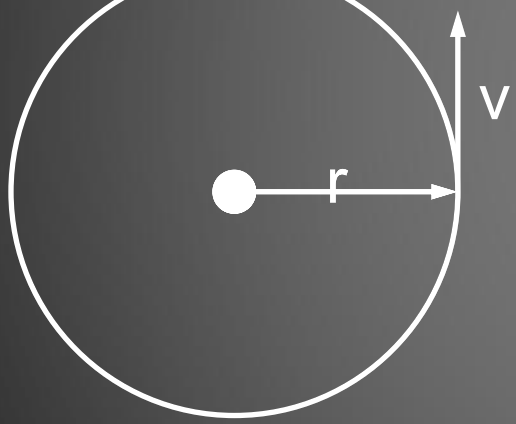
- Circular
- Equatorial (inclination is zero)
- Radius is such that angular rate of orbit is the same as the angular rate of the Earth
  - Many practical uses
- Radius is about 42,000 km, period of 23 hrs, 56 min

GOES-8 Typical FOV



Example 1

A s/c is placed in areosynchronous orbit above Mars (1 Mars day = 24.6hrs). Find e, r, h, E, and v



$\mu_{mars} = 4.305 \times 10^4$ km^3/s^2

An areosynchronous orbit is defined as a circular orbit of period = 1 Mars day. Therefore,

$e = \underline{0}$

$P = 24.6(3600) = 8.856 \times 10^4$ sec

$8.856 \times 10^4 \text{ sec} = 2\pi r^{3/2}/\text{sqrt}(\mu)$

$r = \underline{2.045 \times 10^4 \text{ km}}$

$v = (\mu/r)^{1/2} = \underline{1.45 \text{ km/sec}}$

$h = rv\cos\gamma = rv \quad (\gamma=0)$

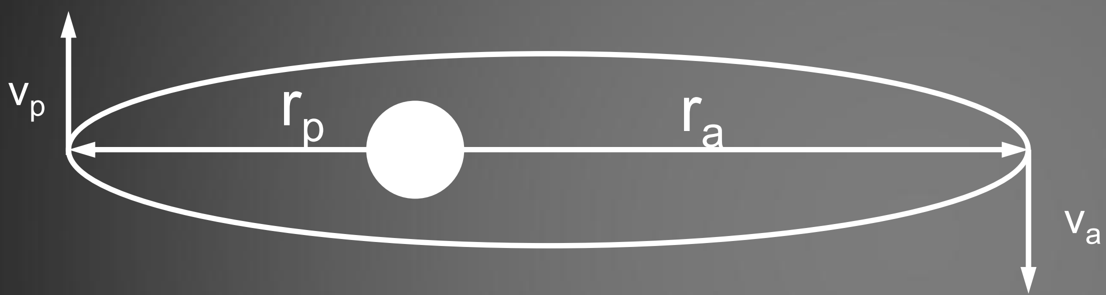
$h = \underline{2.967 \times 10^4 \text{ km}^2\text{/sec}}$

$E = v^2/2 - \mu/r = -\mu/(2a) = -\mu/(2r)$

$E = \underline{-1.05 \text{ km}^2\text{/sec}^2}$

Example 2

A s/c is in orbit around the Earth with $e = 0.1$ and an altitude of perigee of 200 km. Find E, a, r_a, v_a, v_p, and P.



$\mu_E = 3.986 \times 10^{14}\ m^3/s^2$
$r_E = 6.378 \times 10^6$ m

$r_p = r_E + h_p$
$r_p = 6.378 \times 10^6 + 2.0 \times 10^5$
$r_p = 6.578 \times 10^6$ m

$a = r_p/(1-e) = r_a/(1+e)$
$r_p/0.9 = r_a/1.1 \Rightarrow r_a = (1.1/0.9)r_p$
$r_a = \underline{8.04 \times 10^6 \text{ m}}$

$a = r_p/(1-e) = (r_p/0.9)$
$a = \underline{7.309 \times 10^6 \text{ m}}$
$E = v^2/2 - \mu/r = -\mu/(2a)$
$E = -3.986 \times 10^{14}/1.4618 \times 10^7$
$E = \underline{-2.727 \times 10^7 \text{ m}^2/\text{sec}^2}$

Example 2 (cont.)

$$\frac{v^2}{2} = -2.727\times10^7 + \frac{\mu}{r}$$

At periapsis,

$$v_p = \left(-5.4536\times10^7 + 2\left(\frac{3.986\times10^{14}}{6.578\times10^6}\right)\right)^{1/2}$$

$$\boxed{v_p = 8.164\times10^3\,\text{m/s}}$$

At apoapsis,

$$v_a = \left(-5.4536\times10^7 + 2\left(\frac{3.986\times10^{14}}{8.04\times10^6}\right)\right)^{1/2}$$

$$\boxed{v_a = 6.680\times10^3\,\text{m/s}}$$

And the period is,

$$P = 2\pi\left(\frac{a^3}{\mu}\right)^{1/2} = 2\pi\left(\frac{(7.309\times10^6)^3}{3.986\times10^{14}}\right)^{1/2}$$

$$\boxed{P = 6.219\times10^3\,\text{sec} = 1.73\ \text{hrs}}$$

Parabolic and Hyperbolic Orbits

Parabola

- Possible orbit path, although rarely found in nature.
- Borderline case between closed and open orbits.
  - So, defines escape conditions
- For all conic sections:

$$r_p = p/(1+e)$$

$$r_a = p/(1-e)$$

- For a parabola, since e = 1:

$$r_p = p/2$$

$$r_a = \infty$$

Escape Speed

- Defined as the velocity required such that the s/c will reach ∞ at 0 speed
  - Borderline between open and closed orbits
  - Parabola
- $E = v^2/2 - \mu/r = \text{constant}$
- At ∞, $E = 0 \Rightarrow -\mu/(2a) = 0$, $a = \infty \Rightarrow$ parabola
- So, to calculate the velocity required to escape a body's gravitational force

$$\frac{{v_{esc}}^2}{2} = -\frac{\mu}{r} = 0$$

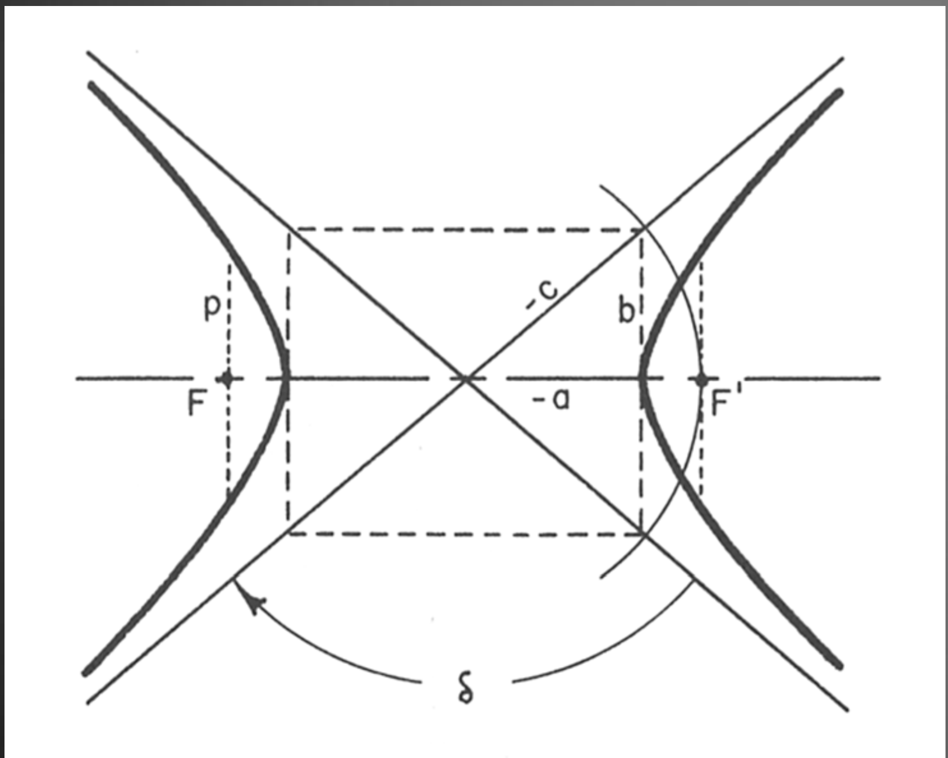
$$\boxed{{v_{esc}}^2 = \sqrt{\frac{2\mu}{r}}} \qquad (25)$$

Independent of direction

Note that: $\boxed{{v_{esc}}^2 = \sqrt{2}\, v_{circ}}$ and that as r increases, v_{esc} decreases

Hyperbolic Orbits

A hyperbolic path is required when we define an orbit that reaches infinity with some nonzero velocity. Examples include meteors and Mars probes.



$$\sin\frac{\delta}{2} = \frac{a}{c}$$ where δ = angle between asymptotes (turn angle)

From (14) $$e = \frac{c}{a}$$

So,

$$\boxed{\sin\frac{\delta}{2} = \frac{1}{e}} \quad (26)$$

As e increases, δ decreases

Hyperbolic Orbits (cont.)

If at a position r, $v=v_{esc}$,
$v \to 0$ as $r \to \infty$

If, however, at the same position r, $v > v_{esc}$
$v \to v_\infty$ where $v_\infty > 0$ as $r \to \infty$

Here, v_∞ is called the hyperbolic excess speed or v-infinity (v_{inf} or v_∞) where,

$$E = \frac{v^2}{2} - \frac{\mu}{r} = \frac{v_\infty^{\ 2}}{2} - \overbrace{\frac{\mu}{r_\infty}}^{0}$$

$$\boxed{v_\infty = \left[v^2 - \frac{2\mu}{r}\right]^{1/2}} \quad (27)$$

$$v_\infty = \left[v^2 - v_{esc}^{\ 2}\right]^{1/2}$$

Example 3

A planetary probe is passing by the Earth such that γ = 0 at

$$r = 7.315 \text{ x } 10^6 \text{ m}$$

$$v = 1.189 \text{ x } 10^4 \text{ m/s}$$

What type of orbit is this? Find E, e, a

Since γ = 0, this is either periapsis or apoapsis.

$$E = \frac{v^2}{2} - \frac{\mu}{r} = \frac{\left(1.139 \times 10^4\right)^2}{2} - \frac{3.936 \times 10^{14}}{7.315 \times 10^6}$$

$$\boxed{E = 1.620 \times 10^7 \frac{m^2}{s^2}}$$

Since E>0, this is an open orbit => hyperbola so,

$$r = r_p = 7.315\text{x}10^6 \text{ m}$$

Example 3 (cont.)

$$E = -\frac{\mu}{2a} \Rightarrow a = -\frac{\mu}{2E} = \frac{-3.936\times10^{14}}{2\left(1.620\times10^{7}\right)}$$

$$\boxed{a = -1.23\times10^{7}\,\mathrm{m}}$$ Note: Negative Sign

$$r_p = a\left(1-e\right)E$$

$$\Rightarrow e = \left(1-\frac{r_p}{a}\right) = 1-\left(\frac{7.315\times10^{6}}{-1.32\times10^{7}}\right)$$

$$\boxed{e = 1.595}$$ Note: e>1

Note that this is an escape or open orbit. What will its residual velocity be when it completely breaks free of Earth's gravity?

v at r = ∞ => v_∞

$E = v_\infty/2 = 1.620 \times 10^7\ m^2/s^2$

$v_\infty = \underline{5.692 \times 10^3\ m}$

Classical Orbital Elements

Planar Motion

For planar (2-D) conic sections, the shape and size of an orbit are determined by 2 parameters

e: eccentricity, defines the type of conic orbit

a: semi-major axis, defines the size of a conic orbit

Note that these 2 parameters could also be determined from knowledge of (E and e) or (E and h)

A 3rd parameter is required to determine the position of the s/c along the orbit

θ: true anomaly

And a 4th parameter is required to determine the angular orientation of the orbit within the 2D plane.

In general (3-D) space, 6 orbital elements are required to uniquely define the size, shape and orientation of an orbit as well as the s/c position within that orbit. This is equivalent to the 6 components of the r and v vectors required to uniquely specify a s/c state.

Classical Orbit Elements

The most common set of orbital elements used is:

a, semi-major axis, a constant defining the size of the orbit.

e, eccentricity, a constant defining the type/shape of the orbit

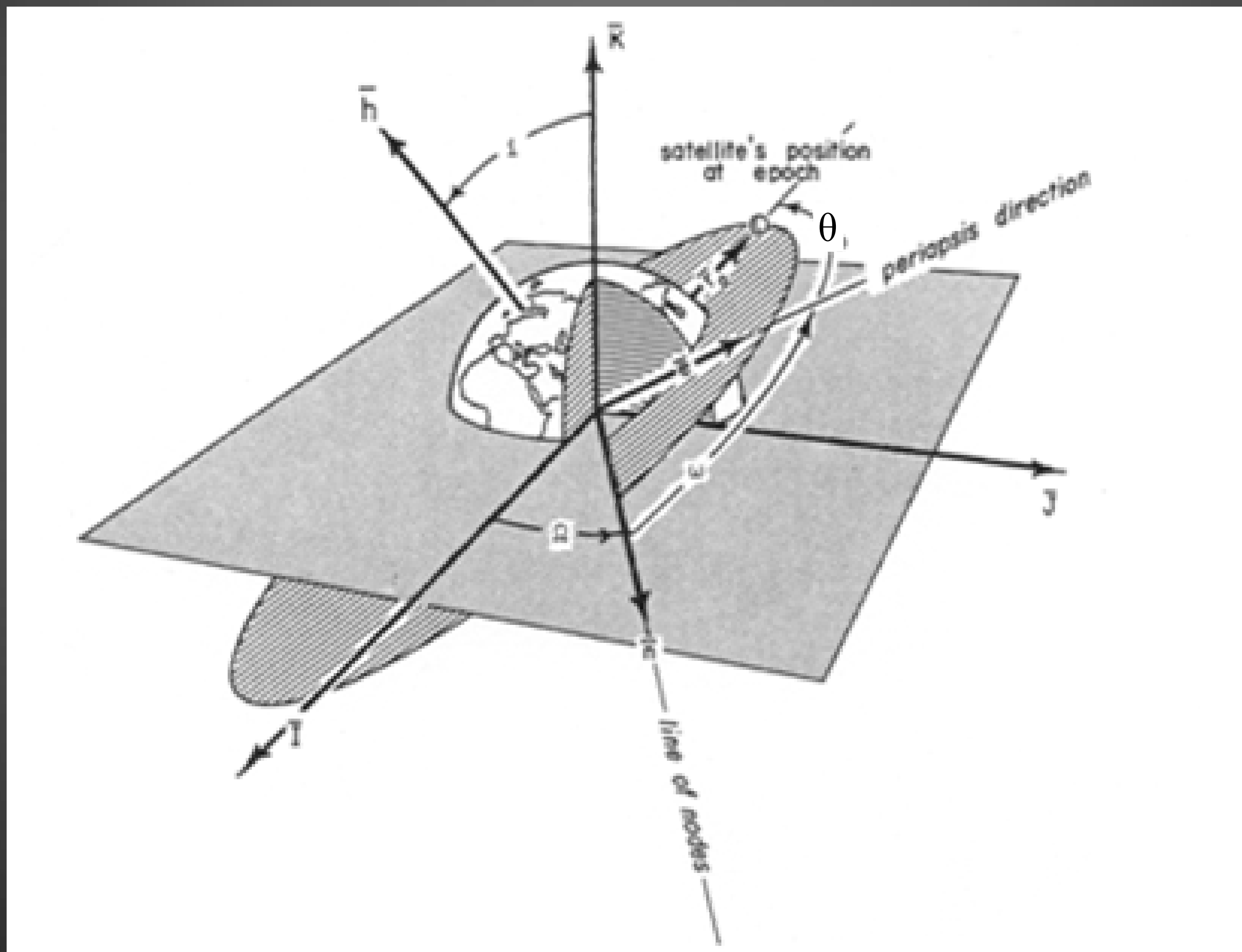
i, inclination, the angle between the z axis and the angular momentum vector

Ω, longitude of ascending node, the angle in the xy plane between the x axis, and the position where the s/c crosses the xy plane in a +z direction. Measured ccw when view from the +z side of the xy plane

w, argument of periapsis, the angle in the plane of the s/c orbit ($x_w y_w$) between the ascending node and periapsis. Measured in the direction of s/c motion

θ, true anomaly, the angle in the plane of the s/c orbit (x_w, y_w) between the periapsis and the s/c position. Measured in the direction of s/c motion.

Graphical Representation



Canonical Units

To simplify calculations a canonical reference unit system is sometimes employed. For problems in which the Earth is the primary gravitational body we define:

MU: 1 mass unit = mass of Earth

DU: 1 distance unit = radius of the Earth (6378 km)

TU: 1 time unit such that the speed of a spacecraft in a 1 DU circular orbit about the Earth is 1 DU/TU

$$1\ \text{TU} = 1\text{DU}/v_c \ (13.447\ \text{minutes})$$

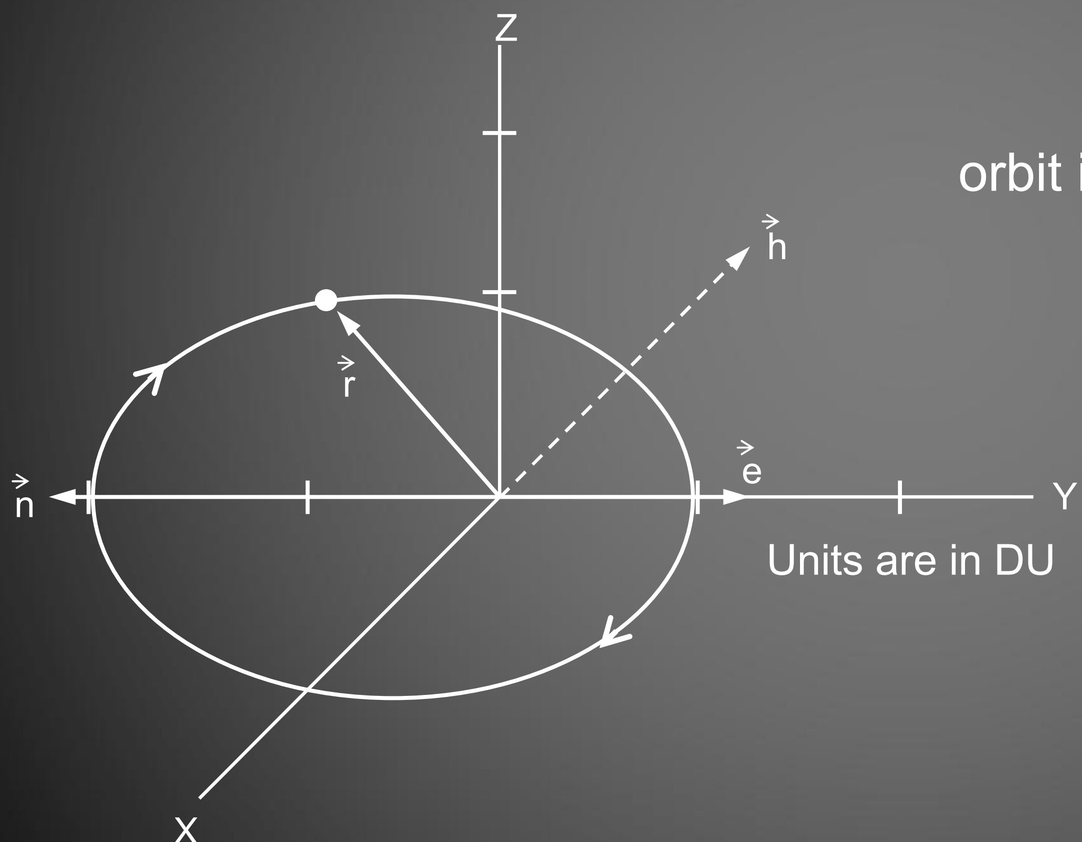
$$\text{where } v_c = 7.905\ \text{km/s}$$

When using such a convention:

$$\mu = 1\ \text{DU}^3/\text{TU}^2$$

Example 4

By inspection determine the canonical orbital elements for the following orbit.



orbit is drawn in yz plane

Units are in DU

Example 4 (cont.)

$$\vec{r} = [0,-1,1]$$

$$2a = 3\text{DU}$$

$$\boxed{a = 1.5\ \text{DU}}$$

$$r_p = 1\text{DU} = a(1-e)$$

$$1 = 1.5(1-e)$$

$$\frac{2}{3} = 1 - e$$

$$\boxed{e = 0.33}$$

$$\vec{h} \perp \hat{z} \Rightarrow \boxed{i = 90^{o}}$$

$$\boxed{\begin{aligned} \Omega &= 270^{o} \\ w &= 180^{o} \\ \theta &= 225^{o} \end{aligned}}$$