Adaptive Control with a Nested Saturation Reference Model

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This paper introduces a neural network based model reference adaptive control architecture that allows adaptation in the presence of saturation. The given plant is approximately feedback linearized, with adaptation used to cancel any matched uncertainty. A nested saturation based reference model is used. This law allows the incorporation of magnitude actuator saturation and has useful small gain properties. Depending on the bandwidth and saturation limits, the reference model based on this law eases off on the aggressiveness of the desired trajectory thus avoiding saturation. However, actuator saturation might yet occur due to uncertainty or external disturbances. In order to protect the adaptive element from such plant input characteristics, the nested saturation reference model is augmented with a pseudo-control hedging signal that removes these characteristics from the adaptive element's training signal.

Nomenclature

b_v, b_w	neural network biases
Δ	function approximation error
δ	control vector / actuator deflections
e	tracking error $x_r - x$
e_{cr}	ref. model command tracking error $x_c - x_r$
e_c	command tracking error $x_c - x$
K	linear compensator gains
$f, \hat{f}$	actual, estimated plant dynamics
$g, \hat{g}$	actual, estimated actuator dynamics
NN	neural network
ν	pseudo-control vector
PCH	pseudo-control hedging
V, W	neural network input, output weights
x	state vector

Subscripts

ad	adaptive signal
c	commanded
des	desired
h	hedge
lc	linear compensator
r	reference model

Introduction

Neural Network (NN) based direct adaptive control has recently emerged as an enabling technology for practical flight control systems. This particular architecture has been applied in simulation to various applications such as the X-33 Reusable Launch Vehicle,[1] tilt-rotor[2] and transport aircraft.[3] This architecture has also been flight-tested on a range of flight vehicles such as the X-36 Tailless fighter,[4] JDAM guided munitions[5] and on an unmanned helicopter.[6]

In implementing this architecture, initial applications assumed no actuator (input) saturation. If input saturation is encountered, the adaptive element (a neural network) incorrectly adapts to these input characteristics. In order to overcome this problem the reference model is modified in a specific way in order to remove input characteristics from the training signal of the neural network. This method called pseduo-control hedging (PCH) was developed initially to adaptively control the attitude dynamics of the X-33.[7] Apart from input saturation pseduo-control hedging may also be used to completely remove any input characteristics the control designer does not want the adaptive element to see. Such characteristics might include actuator magnitude saturation, actuator rate limits, latency and other linear input dynamics. In addition to enabling continued adaptation in the presence of input dynamics, it was shown that a proven domain of attraction for the closed loop system is at least as large as that without PCH.

It has been shown that as long as the external command and the *isolated nonadaptive system* states are close, boundedness of the reference model, plant and neural network states may be shown for certain amounts of saturation and consequently certain amounts of hedging.[7] In present work that uses this architecture,[1,4] linear reference models are used. In general, PCH modifies the reference model dynamics and thus poses a problem when large external commands are applied to a linear reference model. For example in the position control of a vehicle a large position command causes a linear reference model to immediately saturate the controls until the plant state is close to command. The problem was partially tackled by introducing a nonlinear reference model contain-

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ing limits on the maximum speed that could be used to achieve a large position command.[6] This method however has problems where the poles of the reference model change when these limits are active.

In this paper the use of a nested saturation based reference model[8] is proposed. For a certain class of systems in feedforward form,[9] saturation elements may be used to stabilize the system with bounded control guaranteeing Global Asymptotic Stability (GAS) and Local Exponential Stability (LES). This nested saturation law was first introduced to stabilize a chain of integrators[8] and generalized by Sontag.[10] A perfectly feedback linearized system is set of *n-integrators*. A natural choice of reference model is a set of n-integrators controlled using the nested saturation law which takes into account magnitude bounded actuator input and reflects the structure of a feedback linearized plant with magnitude saturation. Adaptation is used to cancel any matched uncertainty in the approximately linearized system, and a nested saturation based reference model is used to generate the trajectory. PCH is used to protect the neural network from incorrect adaptation when large uncertainty or external disturbances cause actuator magnitude saturation. The combined approach is expected to entail no need to avoid actuator saturation or large external commands.

First, the adaptive control architecture is introduced along with PCH. A discussion on the choice of reference model is made by comparing linear and nonlinear reference models, followed by the nested saturation law. Finally, the architecture is applied to a 4^{th} order plant and simulation results are presented.

Controller

Consider the following nonlinear system in first order form

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots \qquad (1)$$
$$\dot{x}_n = f(x, \delta) \qquad (2)$$
$$\delta = g(x, \delta_{des}) \qquad (3)$$

where $x \in \mathcal{R}^n$, x_i, $i = 1, \cdots, n$ being the elements of x, $\delta \in \mathcal{R}$. Here, f represents the plant dynamics and g the state-dependent actuation nonlinearity. Herein, δ_{des} is the desired actuator (control) deflection while δ is the actual deflection. Typically, g represents actuator magnitude saturation. Both f, g are assumed to be only approximately known. The objective is to track a bounded external command $x_c \in \mathcal{R}^n$, while protecting the adaptive element from attempted correction of actuator input characteristic g.

An approximate model for the dynamics may be introduced as

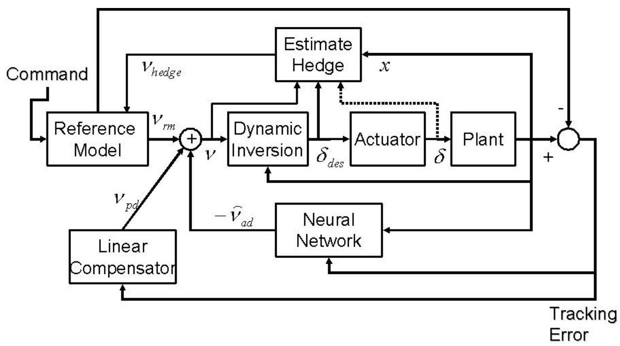

Fig. 1 Model Reference Adaptive Control Architecture with Pseudo-control Hedging

$$\nu \quad = \quad \hat{f}(x, \delta_{des}) \qquad (4)$$

where, ν is the desired pseudocontrol. For example, in the case of second order position control of mechanical systems it represents the desired acceleration. A restriction on $\hat{f}$ is that it should be of a form that allows one to formulate the dynamic inverse as

$$\delta_{des} = \hat{f}^{-1}(x, \nu) \qquad (5)$$

and δ_{des} is the actuator deflection that we expect will achieve the desired pseudocontrol. In introducing these approximate models and formulation of the controller, it is assumed that the full state x, is available for feedback. Output feedback formulations of this architecture are also available.[11]

Substituting the inverse dynamics Eq. (5) into Eq. (2) results in the following approximately linearized system

$$\dot{x}_n = \nu + \Delta(x, \delta) \qquad (6)$$

ν is the desired pseudocontrol that as yet to be designed. The model error function is composed of

$$\Delta(x, \delta) = f(x, \delta) - \hat{f}(x, \delta) \qquad (7)$$

Design, ν to be of the form

$$\nu = \nu_{cr} + \nu_{lc} - \hat{\nu}_{ad} \qquad (8)$$

where ν_{cr} is the output of a reference model, ν_{lc} is the output of a compensator that stabilizes the linearized dynamics and $\hat{\nu}_{ad}$, the output of an adaptive element such as a neural network that is designed to cancel the effects of model error Δ. If only regulation is required, ν_{cr} may be dropped resulting in $\nu = \nu_{lc} - \hat{\nu}_{ad}$. This architecture is illustrated in Figure 1.

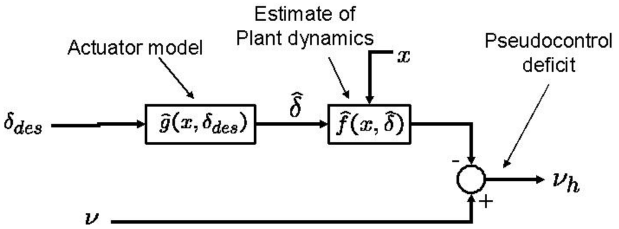

Fig. 2 Pseudocontrol Hedge signal calculation

Pseudo-control Hedging

Normally, for a system in first order form, the reference model dynamics may be designed as

$$\begin{aligned}\dot{x}_{r_1} &= x_{r_2}\\ \dot{x}_{r_2} &= x_{r_3}\\ &\vdots\\ \dot{x}_{r_n} &= \nu_{cr}(x_c, x_r)\end{aligned} \tag{9}$$

where $x_r \in \mathcal{R}^n$ are the states of the reference model and $x_c \in \mathcal{R}^n$ a bounded external command signal. This form however, does not account for actuator dynamics. If the actuators are saturated, the reference model will continue to demand tracking as though full authority were still available. This results in the adaptive element attempting to adapt to the input nonlinearity. Pseudo-control Hedging (PCH) is used to protect the adaptive element from such input characteristics. One way to describe the PCH method is: *move the reference model in the opposite direction (hedge) by an estimate of the amount the plant did not move due to system characteristics the control designer does not want the adaptive element to see.*[7] This will prevent the system characteristic from appearing in the model tracking error dynamics to be developed in the sequel. An approximate model of the actuator input characteristic $\hat{g}$ is introduced and is used formulate the PCH signal that removes any actuator characteristics from the tracking error dynamics. This may be represented as

$$\hat{\delta} \quad = \quad \hat{g}(x, \delta_{des}) \tag{10}$$

If in fact the actuator position signals are also available then, $\hat{g}$ is not required and

$$\hat{\delta} = \delta$$

The PCH signal is the difference between the commanded and achieved pseudocontrol

$$\begin{aligned}\nu_h &= \hat{f}(x, \delta_{des}) - \hat{f}(x, \hat{\delta})\\ &= \nu - \hat{f}(x, \hat{\delta})\end{aligned} \tag{11}$$

The *estimate hedge* block of Figure 1 is given by Eq. (11) and illustrated in Figure 2. Eq. (9) may now be augmented with the hedging signal resulting in the removal of the actuator characteristic from the tracking error dynamics. The reference model dynamics now include the hedging term

$$\dot{x}_{r_n} = \nu_{cr}(x_c, x_r) - \nu_h \tag{12}$$

Notice here that the hedge signal affects the reference model output ν_{cr} only through changes in reference model dynamics and that the instantaneous pseudocontrol output of the reference model in not changed by the use of PCH and remains

$$\nu_{cr} = f_r(x_c, x_r) \tag{13}$$

Tracking Error Dynamics

Defining the reference model tracking error as

$$e \triangleq x_r - x \tag{14}$$

its dynamics may be found by directly differentiating Eq. (14)

$$\dot{e} = \begin{bmatrix} x_{r_2} - x_2 \\ \vdots \\ \dot{x}_{r_n} - \dot{x}_n \end{bmatrix} \tag{15}$$

Considering $\dot{e}_n$,

$$\begin{aligned}\dot{e}_n &= \dot{x}_{r_n} - \dot{x}_n\\ &= \nu_{cr} - \nu_h - f(x, \delta)\\ &= \nu_{cr} - \nu + \hat{f}(x, \hat{\delta}) - f(x, \delta)\\ &= -\nu_{lc} + \hat{\nu}_{ad} + \hat{f}(x, \hat{\delta}) - f(x, \delta)\\ &= -\nu_{lc} - (\Delta(x, \delta, \hat{\delta}) - \hat{\nu}_{ad})\end{aligned} \tag{16}$$

If ν_{lc} is chosen to be a linear compensator of the form

$$\nu_{lc} = \begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix} e \tag{17}$$

Hence, the overall tracking error dynamics may now be expressed as

$$\dot{e} = Ae + B\left[\hat{\nu}_{ad} - \Delta(x, \delta, \hat{\delta})\right] \tag{18}$$

where,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \\ -K_1 & -K_2 & \cdots & \cdots & -K_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \tag{19}$$

The compensator gains are chosen such that A is Hurwitz.

It now remains for $\hat{\nu}_{ad}$ to be designed to cancel the model error $\Delta(x, \delta, \hat{\delta})$ and minimize the forcing term in Eq. (18). Hence $\hat{\nu}_{ad} = \hat{\nu}_{ad}(x, \delta, \hat{\delta})$ to effectively cancel Δ. However δ, the actuator position is not available in order to design the adaptive term $\hat{\nu}_{ad}$. Hence we make the following assumption.

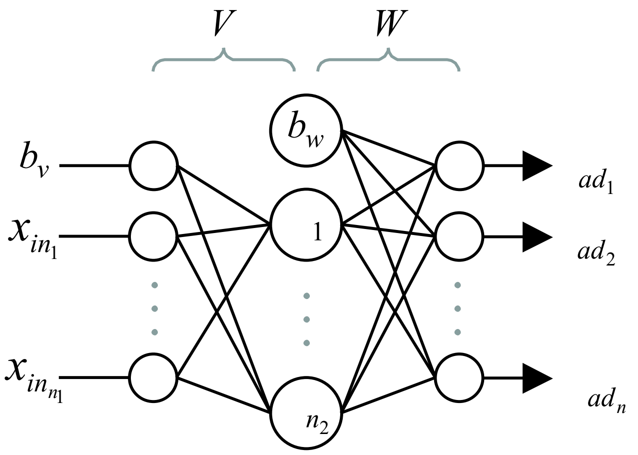

Fig. 3 Neural Network with one hidden layer

Assumption 1. *The actual actuator position can be expressed as*

$$\delta = \delta(x, \hat{\delta})$$

With this assumption, one may represent the tracking error dynamics as

$$\dot{e} = Ae + B\left[\hat{\nu}_{ad}(x, \hat{\delta}) - \Delta(x, \hat{\delta})\right] \tag{20}$$

where $\hat{\nu}_{ad}$ is now only required to be dependent on available information.

Adaptive Element

Single hidden layer perceptron Neural Networks (NNs) are universal approximators.[12] Hence, given a sufficient number of hidden layer neurons and appropriate inputs, it is possible to train the network to cancel model error. Figure 3 shows the structure of a single hidden layer network whose input-output map may be expressed as

$$\nu_{ad_k} = b_w \theta_{wk} + \sum_{j=1}^{n_2} w_{jk} \sigma_j(z_j) \tag{21}$$

where, $k = 1, ..., n_3$ and,

$$\sigma_j(z_j) = \sigma(b_v \theta_{vj} + \sum_{i=1}^{n_1} v_{ij} x_{in_i}) \tag{22}$$

Here, n_1, n_2 and n_3 are the number of inputs, hidden layer neurons and outputs respectively. $x_{in_i}, i = 1..n_1$, denote the inputs to the NN. The scalar σ_j is a sigmoidal activation activation function,

$$\sigma(z) = \frac{1}{1 + e^{-az}} \tag{23}$$

where, a is the so called *activation potential.* For convenience, define the following weight matrices

$$V = \begin{bmatrix} \theta_{v,1} & \cdots & \theta_{v,n_2} \\ v_{1,1} & \cdots & v_{1,n_2} \\ \vdots & \ddots & \vdots \\ v_{n_1,1} & \cdots & v_{n_1,n_2} \end{bmatrix} \tag{24}$$

$$W = \begin{bmatrix} \theta_{w,1} & \cdots & \theta_{w,n_3} \\ w_{1,1} & \cdots & w_{1,n_3} \\ \vdots & \ddots & \vdots \\ w_{n_2,1} & \cdots & w_{n_2,n_3} \end{bmatrix} \tag{25}$$

$$Z = \begin{bmatrix} V & 0 \\ 0 & W \end{bmatrix} \tag{26}$$

Additionally, define the $\sigma(z)$ vector as

$$\sigma^T(z) = \begin{bmatrix} b_w & \sigma(z_1) & \cdots & \sigma(z_{n_2}) \end{bmatrix} \tag{27}$$

where $b_w > 0$ allows for the threshold, θ_w, to be included in the weight matrix W. Also, $z = V^T \bar{x}$, where,

$$\bar{x}^T = \begin{bmatrix} b_v & x_{in}^T \end{bmatrix} \tag{28}$$

where, $b_v > 0$, is an input bias that allows for thresholds θ_v to be included in the weight matrix V. The input-output map of the SHL network may now be written in concise form as

$$\nu_{ad} = W^T \sigma(V^T \bar{x}) \tag{29}$$

The NN may be used to approximate a nonlinear function, such as $\Delta(.)$. The universal approximation property[12] of NN's ensures that given an $\bar{\epsilon} > 0$, then $\forall$ $\bar{x} \in \mathcal{D}$, where $\mathcal{D}$ is a compact set, $\exists$ an $\bar{n}_2$ and an *ideal* set of weights (V^*, W^*), that brings the output of the NN to within an ϵ-neighbourhood of the function approximation error. This ϵ is bounded by $\bar{\epsilon}$ which is defined by

$$\bar{\epsilon} = \sup_{\bar{x} \in \mathcal{D}} \left\| W^T \sigma(V^T \bar{x}) - \Delta(\bar{x}) \right\| \tag{30}$$

The weights, (V^*, W^*) may be viewed as optimal values of (V, W) in the sense that they minimize $\bar{\epsilon}$ on $\mathcal{D}$. These values are not necessarily unique. The universal approximation property thus implies that if the NN inputs x_{in} are chosen to reflect the functional dependency of $\Delta(\cdot)$, then $\bar{\epsilon}$ may be made arbitrarily small given a sufficient number of hidden layer neurons, n_2.

Boundedness

Associated with the tracking error dynamics given in Eq. (18), is the Lyapunov function.

$$A^T P + PA + Q = 0 \tag{31}$$

Choosing

$$Q > 0 \tag{32}$$

results in a positive definite solution for P.

Assumption 2. *The norm of the ideal weights (V^*, W^*) is bounded by a known positive value.*

$$0 < \|Z^*\|_F \leq \bar{Z} \tag{33}$$

where, $\|\cdot\|_F$ denotes the Frobenius norm. Individually, the ideal layer weights are bounded as $\|W^*\|_F \leq \bar{W}$ and $\|V^*\|_F \leq \bar{V}$

Assumption 3. *The external command x_c is bounded.*

$$\|x_c\| \leq \bar{x}_c \tag{34}$$

Assumption 4. *The states of the reference model, remain bounded for permissable plant and actuator dynamics.*

Assumption 5. *Note that, Δ depends on ν_{ad} through ν, whereas ν_{ad} has to be designed to cancel Δ. Hence the existence and uniqueness of a fixed-point-solution for $\nu_{ad} = \Delta(x, \nu_{ad})$ must be assured through selection of $\hat{f}$. A sufficient condition is to ascertain that the map $\nu_{ad} \mapsto \Delta(x, \nu_{ad})$ is a contraction over the entire input domain of interest, or $\|\partial\Delta/\partial\nu_{ad}\| < 1$. This condition is equivalent to the following condition on $\hat{f}$.*

$$\begin{aligned}\left\|\frac{\partial\Delta}{\partial\nu_{ad}}\right\| &= \left\|\frac{\partial\Delta}{\partial\delta}\frac{\partial\hat{f}^{-1}}{\partial\nu}\frac{\partial\nu}{\partial\nu_{ad}}\right\| \\ &= \left\|\left(\frac{\partial f}{\partial\delta} - \frac{\partial\hat{f}}{\partial\delta}\right)\frac{\partial\hat{f}^{-1}}{\partial\nu}\right\| \\ &= \left\|\frac{\partial f}{\partial\delta}\frac{\partial\hat{f}^{-1}}{\partial\nu} - I\right\| < 1\end{aligned} \tag{35}$$

For a SISO system, condition (35) is equivalent to

$$sgn(\partial f/\partial\delta) = sgn(\partial\hat{f}/\partial\delta) \tag{36}$$

and,

$$|\partial\hat{f}/\partial\delta| > |\partial f/\partial\delta|/2 \tag{37}$$

Condition (36) states that unmodeled control reversal is not permissable and (37) places a lower bound on the estimate of control effectiveness.

Theorem 1. *Consider the system given by (2) together with the inverse law (5) and Assumptions 2,3,4,5 where,*

$$r = (e^T P B)^T \tag{38}$$
$$\hat{\nu}_{ad} = \nu_{ad} + \nu_r \tag{39}$$
$$\nu_{ad} = W^T \sigma(V^T \bar{x}) \tag{40}$$
$$\nu_r = -K_r(\|Z\|_F + \bar{Z}) r \frac{\|e\|}{\|r\|} \tag{41}$$

with $K_r > 0 \in \mathcal{R}$, and where W, V satisfy the adaptation laws

$$\dot{W} = -\left[(\sigma - \sigma' V^T \bar{x}) r^T + \kappa\|e\| W\right]\Gamma_W \tag{42}$$
$$\dot{V} = -\Gamma_V\left[\bar{x}(r^T W^T \sigma') + \kappa\|e\| V\right] \tag{43}$$

with, $\Gamma_W, \Gamma_V > 0$ and $\kappa > 0$, guarantees that reference model tracking error (e) and NN weights (W, V) are uniformly ultimately bounded.

Proof. In the following proof a '*' represents *ideal values*. Define the following variables, $\tilde{W} \triangleq W^* - W, \tilde{V} \triangleq V^* - V, z = V^T\bar{x}, \tilde{z} = z^* - z$. The arguments to the sigmoidal activation function σ are dropped for clarity and conciseness. The Taylor expansion of $\sigma(z)$ around the estimated weights is given by

$$\sigma(z^*) = \sigma(z) + \left.\frac{\partial\sigma(s)}{\partial s}\right|_{s=z}(z^* - z) + \mathcal{O}^2(\tilde{z})$$

The function approximation error may now be expressed as

$$\begin{aligned}\tilde{f} = f - \hat{f} &= W^{*T}\sigma^* - W^T\sigma + \epsilon \\ &= W^{*T}\left[\sigma(z) + \sigma'\tilde{z} + \mathcal{O}^2(\tilde{z})\right] - W^T\sigma + \epsilon \\ &= \tilde{W}^T(\sigma - \sigma' z) + W^T\sigma'\tilde{z} + w\end{aligned} \tag{44}$$

where,

$$w = \tilde{W}^T\sigma' z^* + W^{*T}\mathcal{O}^2(\tilde{z}) + \epsilon \tag{45}$$

noting that $\Delta = W^{*T}\sigma^* + \epsilon$, substituting for ν_{ad} and using Eq. (44) in Eq. (18), the tracking error dynamics may be written as

$$\dot{e} = Ae + B\left[-\left(\tilde{W}^T(\sigma - \sigma' z) + W^T\sigma'\tilde{z} + w\right) + \nu_r\right] \tag{46}$$

By computing bounds on $\nu_{ad}, \bar{x}$ and $\mathcal{O}^2(\tilde{z})$ the disturbance term w may be bounded as

$$\|w\| = c_0 + c_1\|\tilde{Z}\| + c_2\|e\|\|\tilde{Z}\| + c_3\|\tilde{Z}\|^2 \tag{47}$$

where, c_0, c_1, c_2, c_3 are known constants and $\tilde{Z}$ follows from the definition of Eq. (26). A lyapunov candidate function is

$$L(e, \tilde{W}, \tilde{V}) = \frac{1}{2}\left[e^T P e + tr\left(\tilde{W}\Gamma_W^{-1}\tilde{W}^T\right) + tr\left(\tilde{V}^T\Gamma_V^{-1}\tilde{V}\right)\right]$$

Using the weight update equations of Eq. (42) and Eq. (43), the time derivative of L along trajectories can be expressed as

$$\dot{L} = -\frac{1}{2}e^T Q e + r^T(-w + \nu_r) + \kappa\|e\| tr\left(\tilde{Z}^T Z\right)$$

Using $Z = Z^* - \tilde{Z}$ and $\|Z\| \geq \|\tilde{Z}\| - \bar{Z}$ along with the robustifying term of Eq. (41) and requiring that $K_r > c_2$, $\kappa > \|PB\|c_3$, $\dot{L}$ may be bounded as

$$\begin{aligned}\dot{L} &\leq -\frac{1}{2}\lambda_{min}(Q)\|e\|^2 + \left(\|PB\|c_1 + \kappa\bar{Z}\right)\|\tilde{Z}\|\|e\| \\ &\quad - \left(\kappa - \|PB\|c_3\right)\|e\|\|\tilde{Z}\|^2 + c_0\|PB\|\|e\| \\ \dot{L} &\leq -\frac{1}{2}\lambda_{min}(Q)\|e\|^2 - \left(\kappa - \|PB\|c_3\right)\|e\|\|\tilde{Z}\|^2 \\ &\quad + a_0\|e\| + a_1\|e\|\|\tilde{Z}\|\end{aligned}$$

where, defining

$$\begin{aligned}a_0 &= 2\bar{Z}k_2 + \bar{\epsilon} \\ a_1 &= 2\bar{a}\bar{Z}k_0k_1k_2\|PB\| + \kappa\bar{Z}\end{aligned}$$

and,

$$\begin{aligned}k_1 &= (1 + b_w + n_2) \\ k_0 &= b_v + \bar{x}_c + k_1\bar{Z} \\ k_2 &= b_w + n_2\end{aligned}$$

By selecting $\lambda_{min}(Q), \kappa$ and learning rates (Γ_W and Γ_V), $\dot{L} \leq 0$ everywhere outside a compact set that is entirely within the largest level set of L, which in turn lies entirely within the compact set D.[7] Thus for initial conditions within D, the tracking error e, and neural network weights $\tilde{W}, \tilde{V}$ are uniformly ultimately bounded.

□

Reference Model

The pseudo-control build up in Eq. (8) contains the reference model output ν_{cr}. One may now examine the effect of choosing ν_{cr} by considering the isolated nonadaptive subsystem where the tracking error $e = 0$. Assuming $\hat{\nu}_{ad}^*$ is the post-adaptive output of the adaptive element ($W = W^*, V = V^*$). The closed loop system maybe written as

$$\dot{x}_n = f(x, \delta) \tag{48}$$
$$= f(x, g(x, \delta_{des})) \tag{49}$$
$$= f(x, g(x, \hat{f}^{-1}(x, \nu_{cr} + \cancel{\nu_{lc}} - \hat{\nu}_{ad}^*))) \tag{50}$$

where $\nu_{lc} = 0$ because tracking error e is assumed to be 0. If the adaptation is capable of exactly cancelling the model error the dynamics become

$$\dot{x}_n = f(x, g(x, f^{-1}(x, \nu_{cr}))) \tag{51}$$

Additionally, when $\delta \approx \delta_{des}$, the dynamics become

$$\dot{x}_n \approx \nu_{cr} \tag{52}$$

ν_{cr} could be designed so that δ and δ_{des} always match, perhaps as the output of an optimal trajectory generator that takes into account the system dynamics f and actuator input characteristics g. Or, a simple linear design could be selected for ν_{cr}. Additionally, ν_{cr} may also be chosen to facilitate stability analysis of the overall system.

With respect to using the neural network; consider in detail the isolated non-adaptive system. Define the reference model command tracking error to be

$$e_{cr} = x_c - x_r \tag{53}$$

Define the command tracking error to be

$$e_c = x_c - x \tag{54}$$

Note that $e_c = e_{cr} + e$ and when considering the non-adaptive subsystem the reference model tracking error $e = 0$; hence, $e_c = e_{cr}$.

Therefore,

$$\dot{e}_c = \begin{bmatrix} x_{c_2} - x_2 \\ \vdots \\ \dot{x}_{c_n} - \dot{x}_n \end{bmatrix} \tag{55}$$

Substituting the closed loop nonadaptive subsystem,

$$\dot{e}_c = \begin{bmatrix} x_{c_2} - x_2 \\ \vdots \\ -f(x, g(x, \hat{f}^{-1}(x, \nu_{cr} - \hat{\nu}_{ad}^*))) \end{bmatrix} \tag{56}$$

If no actuators are beyond the saturation limit ($\delta = \delta_{des}$), it becomes,

$$\dot{e}_c = \begin{bmatrix} x_{c_2} - x_2 \\ \vdots \\ -f(x, \hat{f}^{-1}(x, \nu_{cr} - \hat{\nu}_{ad}^*)) \end{bmatrix} \tag{57}$$

$$= \begin{bmatrix} x_{c_2} - x_2 \\ \vdots \\ -\nu_{cr} - \epsilon \end{bmatrix} \tag{58}$$

where, ϵ is the instantaneous residual network approximation error corresponding to the idea weights, and $||\epsilon|| < \bar{\epsilon}$. It is shown in[7] that as long as the external command and isolated nonadaptive system states are close, Lyapunov boundedness results are still valid for certain amounts of saturation and hedging. In general when choosing ν_{cr} it is important to mitigate the effects of having large external commands which leads to extended periods of hedging. This may be achieved by introducing limits on evolution of states in the reference model. The discussion on choice of reference models will be carried out in context of the numerical example in the following section. Three different reference models will be examined with emphasis on the nested saturation reference model.

Linear Reference Models

The *linear reference model* given by Eq. (59) is stable but contains no limits, hence for large commands, the response will still be linear thus saturating the control quickly.

$$\nu_{cr} = \begin{bmatrix} K_1 & K_2 & \cdots K_n \end{bmatrix} e_{cr} \tag{59}$$

Nonlinear Reference Model

The *nonlinear reference model* given by Eq. (60) allows one to impose *prescribable* limits on the evolution of the states.

$$\begin{aligned} \nu_{cr} &= \sigma_n \left[K_n \left(e_{cr_n} + \sigma_{n-1} \left(\frac{K_{n-1}}{K_n} (\alpha) \right) \right) \right] \\ \alpha &\triangleq e_{cr_{n-1}} \dots + \sigma_1 \left(\frac{K_1}{K_2} e_{cr_1} \right) \end{aligned} \tag{60}$$

When none of the limit functions σ_i are active, Eq. (60) is the same as Eq. (59). However, the parameters such as the saturation limits for σ_i must be be chosen correctly, and it is possible to choose limits in an ad-hoc manner such that hedging activity is reduced. It is possible that these parameters may be derived from practical limits such as speed, attitude, angular rate and angular acceleration limits that may be prescribed for an air vehicle.[6] The nonlinear reference model however has other disadvantages. For example, consider a second order reference model (ignoring hedging) with desired real poles at $-a_1, -a_2 \in R^1 > 0$. Based on this desired behavior in the linear region, the nonlinear reference model has the following dynamics.

$$\dot{x}_{r_n} = \nu_{cr} = \sigma_2 \left(K_2 (e_{cr_2} + \sigma_1 \left(\frac{K_1}{K_2} e_{cr_1} \right) \right)$$

with $K_1 = a_1 a_2, K_2 = (a_1 + a_2)$. When neither σ_1 or σ_2 is saturated, the characteristic equation becomes

$$s^2 + K_2 s + K_1 = s^2 + (a_1 + a_2)s + a_1 a_2$$

which has roots at $-a_1, -a_2$. Now assume that σ_1 becomes saturated, then the system becomes

$$\dot{x}_{r_n} = K_2 e_{cr_2} \pm M_1$$

where M_1 is the saturation limit for $\sigma_1(\cdot)$. This system has the characteristic equation

$$s(s + K_2) = s(s + a_1 + a_2)$$

where one of the poles has moved to the origin and the second pole has become faster (assuming both $a_1, a_2 > 0$ which is required for stability). This shifting of poles as different elements of the nonlinear reference model saturate is undesirable because these faster poles can lead to excitation of higher-order dynamics.

Nested Saturation Reference Model

Before the nested saturation law based reference model is introduced the following results that illustrate relevant properties of the law are developed. Associated results are available,[13] a subset of which is provided in the appendix without proof.

Lemma 1. *Consider a chain of n-integrators, given by (63), which may be represented as* $\dot{x} = A_x x + B_x u$, *with* $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ *and*

$$A_x = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad B_x = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \tag{61}$$

then there exists a linear transformation $y = T_{yx} x$ *which transforms (63) into* $\dot{y} = A_y y + B_y u$ *where,*

$$A_y = \begin{bmatrix} 0 & a_n & \cdots & \cdots & a_n \\ 0 & 0 & a_{n-1} & \cdots & a_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & a_2 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad B_y = \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ \vdots \\ a_1 \end{bmatrix} \tag{62}$$

and the elements $a_i \in \mathbb{R} \setminus 0$ *with* $i = 1 \dots n$.

Proof. For a proof and explicit characterization of the transformation T_{yx} see.[13] □

Corollary 1 (Pole location[13]). *If the saturators used are linear saturators, and none of the* σ_i *are saturated, the poles of the linearized closed loop system reside at* $\{-a_1, -a_2, \cdots, -a_n\}$. *During periods when the outermost saturated element is the* k^{th} *saturator,* σ_k, *the poles of the resulting closed loop linear system reside at* $\{-a_1, -a_2, \cdots, -a_{n-k}, 0_1, 0_2, \cdots, 0_k\}$.

Lemma 2. *Consider a chain of integrators*

$$\dot{x}_1 = x_2, \cdots, \dot{x}_n = u \tag{63}$$

For the system given by (63). Given any set of positive constants $\{(L_i, M_i)\}$, *where* $L_i \leq M_i$ *for* $i = 1, \cdots, n$ *and* $M_i < \frac{1}{2} L_{i+1}$ *for* $i = 1, \cdots, n-1$, *and for any set of functions* $\{\sigma_i\}$ *that are* linear saturations *for* $\{(L_i, M_i)\}$, *there exists a linear coordinate transformation* $y = T_{yx} x$ *such that the bounded control*

$$u = -\sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1))) \tag{64}$$

results in a globally asymptotically stable system.

Corollary 2 (Restricted Tracking). *Consider a nonlinear system with magnitude saturation at the input u given by*

$$\dot{x}_1 = x_2, \cdots, \dot{x}_n = \sigma_{n+1}(u) \tag{65}$$

and a compatible reference signal given by

$$\begin{bmatrix} x_d(t), & \dot{x}_d(t), & \cdots & x_d^{(n)}(t) \end{bmatrix} \tag{66}$$

If $|x_d^{(n)}(t)| \leq L_{n+1} - \epsilon$ *for all* $t \geq t_0$ *and for some* $\epsilon > 0$ *and given* linear saturation *functions* σ_i *with parameters* (L_i, M_i) *satisfying,*

$$\begin{aligned} L_i &\leq M_i & i &= 1, \cdots, n+1 \\ M_i &< \frac{1}{2} L_{i+1} & i &= 1, \cdots, n-1 \\ M_n &\leq \epsilon \end{aligned}$$

then, the feedback

$$u = x_d^{(n)} - \sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1)))$$

with $y = T_{yx}e$ *given by Lemma 1, where,* $e_i = x_i - x_d^{(i-1)}$ *for* $i = 1 \cdots n$*, results in a globally asymptotically stable system. Additionally if linear saturators elements are used, the error dynamics are governed by Corollary 1.*

The nested saturation law based reference model may be written as

$$\begin{aligned} \dot{x}_{r_n} &= \nu_{cr} - \nu_h \\ &= \sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1))) - \nu_h \end{aligned} \quad (67)$$

where $y_i = T_{yx}e_{cr}$ where T_{yx} is chosen according to Lemma 1.[13] The limit parameters for σ_i are no longer arbitrary but must be chosen according to Corollary 2. The rate of evolution[13] of the states now take a fixed value when the corresponding saturation element is saturated.

Assumption 6.

$$\lim_{t \to \infty} ||\nu_h|| = 0$$

When the plant states $x(t)$ are such that it does not leave the command-controllable region corresponding to the command x_c, and x_c is such that the system does not need to be saturated in order to maintain x_c then for a range of plant dynamics f, and actuator capabilities, after a finite time $T > 0$ the system trajectory will enter a region close to $e = x_r - x = 0$ where no saturation occurs.

Theorem 2. *Consider the system given by (2) together with the inverse law (5), reference model given by Eq. (67) with gains chosen according to Lemma 1, limits according to Corollary 2 and Assumptions 2,3,4,5,6 with neural network training laws given by Eq. (42), and Eq. (43) with* $K_r, \kappa, \Gamma_V, \Gamma_W$ *chosen in Theorem 1, then the plant states (x) neural network weights (W, V) and reference model states* x_r *are uniformly ultimately bounded.*

Proof. From Assumption 6 there exists a finite time $T > 0$ such that

$$\|\nu_h\| \leq \delta' \qquad \forall t > T, \delta' > 0$$

It can be shown[9] that with ν_h acting as the disturbing input, Eq. (12) is zero-input locally exponentially stable and x_{cr} satisfies an asymptotic bound with linear gain and a non-zero restriction for ν_h (See Definition 5). This implies x_r is bounded and from Theorem 1 with boundedness of e, V, W, boundedness of the plant states x follows. □

Numerical Example

Consider the following system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \delta + 1.8 + x_1 x_2 \delta \\ \delta &= \sigma_5(\delta_{des}) \end{aligned} \quad (68)$$

where σ_5 is a magnitude saturation function with limits (-2,2). The poles of the linear compensator and reference model are chosen such that the closed loop system linearized at the origin has poles at $\{-0.5, -1, -2, -3\}$. Hence, defining the errors as given by Eq. (14),

$$\nu_{lc} = Ke = \begin{bmatrix} 3 & 11 & 14 & 6 \end{bmatrix} e \quad (69)$$

Additionally, the nested saturation reference model output is chosen as

$$\nu_{cr} = \sigma_5(x_c^{(4)} - \sigma_4(y_4 + \sigma_3(y_3 + \sigma_2(y_2 + \sigma_1(y_1))))) \quad (70)$$

where, y is given by Lemma 1 and may be expressed as

$$y = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} x \quad (71)$$

The saturation function, $\sigma_i(L_i, M_i)$ parameters were chosen as

$$\begin{aligned} L_5 &= 2 & M_5 &= L_5 - \bar{\epsilon} \\ \epsilon &= \frac{1}{2} L_5 \\ L_4 &= \epsilon & M_4 &= \epsilon - \bar{\epsilon} \\ L_3 &= \frac{1}{2} L_4 & M_3 &= L_3 - \bar{\epsilon} \\ L_2 &= \frac{1}{2} L_3 & M_2 &= L_2 - \bar{\epsilon} \\ L_1 &= \frac{1}{2} L_2 & M_1 &= L_1 - \bar{\epsilon} \end{aligned}$$

where $\bar{\epsilon}$ is a small positive number used to satisfy a strict inequality. Additionally, $|x_c^{(4)}| < L_5 - \epsilon$. Assuming that the approximate model for the dynamics is given by

$$\nu = \hat{f}(x, \delta_{des}) = \delta_{des} \quad (72)$$

The desired control is simply

$$\delta_{des} = \nu = \nu_{cr} + \nu_{lc} - \hat{\nu}_{ad} \tag{73}$$

where the adaptive element output $\hat{\nu}_{ad}$ is used to cancel the uncertainty $\Delta(x, \delta) = 1.8 + x_1 x_2 \delta$. If it is assumed that actuator position is known, our estimate of actuator position $\hat{\delta} = \delta$ and hence, the pseudo-control hedgingsignal, is given by

$$\nu_h = \nu - \hat{f}(x, \hat{\delta}) = \nu - \delta \tag{74}$$

The neural network to approximate the uncertainty was chosen to have 5 input neurons with inputs $x_{in}^T = [e^T PB, x^T, \delta]$, 3 hidden neurons and learning rates $\Gamma_V = \Gamma_W = 20$. Although the reference model parameters are chosen to avoid actuator saturation, external disturbances and uncertainty may cause the actuator to saturate. It is in these situations of saturation that hedging is required to protect the adaptive element from incorrect adaptation.

Figure 4 shows the step response of the system given by Eq. (68) when using the linear, nonlinear and nested-saturation reference models. Figure 5 shows the control deflection. The linear and nonlinear reference models are aggressive; however the system states are unbounded. The nested saturation law after some initial saturation and hedging activity during the learning phase for the neural network achieves the step command.

Figure 6 shows the response of the system to a square wave command. Both the raw external command x_{1_c} and the nested saturation reference model output x_{r_1} is shown. The states of the system x_1 tracks x_{r_1}. Figure 7 shows the actuator position. At times when the actuator is saturated, the reference model is hedged (Figure 10) to allow the neural network to continue adapting correctly. Figure 8 presents a plot of the uncertainty over time and the Neural Network's approximation of it. Finally, Figure 9 presents the outputs of the different saturation elements in the nested saturation based reference model.

Conclusion

The results presented in this paper are motivated from previous work on model-reference adaptive control.[6] In a simplified form, exactly full-state feedback linearized systems are essentially n-integrator systems which may then be stabilized using conventional linear, nonlinear or adaptive techniques. The nested saturation law and other related results allow actuator limits to be directly incorporated into the control law for the linearized dynamics. Another way it may be employed is by using an n-integrator reference model controlled by the nested saturation law. This leaves the reference model to account for actuator magnitude saturation. It is the latter case that is presented in this paper. In using a nested saturation law as the reference model,

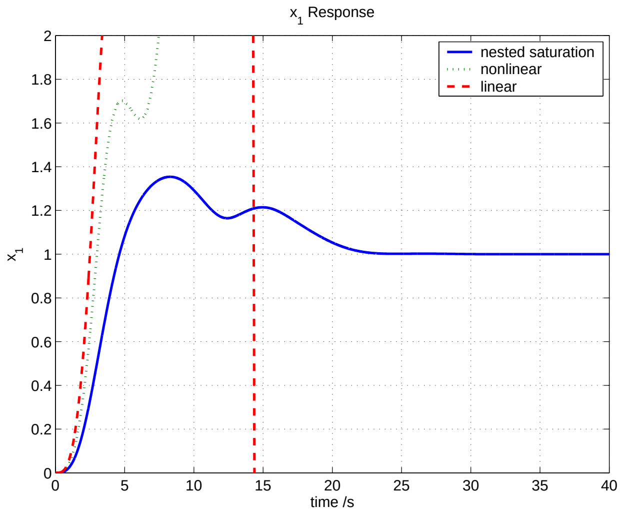

Fig. 4 State x_1 response to a step command for the linear, nonlinear and nested saturation reference models

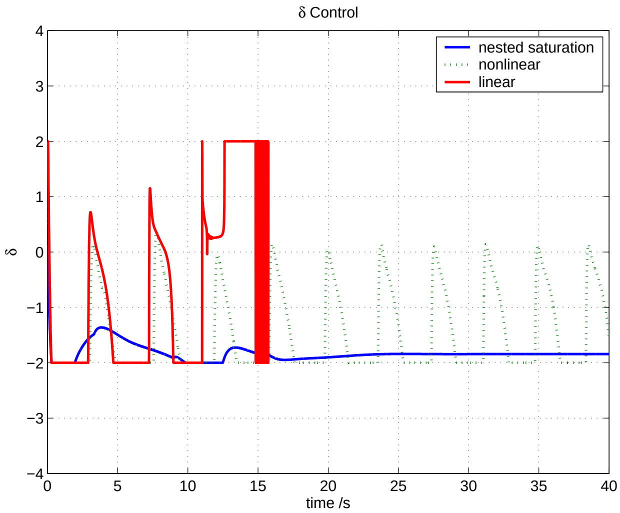

Fig. 5 Control signal δ for the linear, nonlinear and nested saturation reference models during step

problems that arise due to large external commands are mitigated. Another desirable property is that the poles of the system move in a predictable manner, when various elements of the law saturate and is an improvement over the nonlinear reference model[6] with arbitrary saturation functions and limits.

Appendix

Definition 1 (Linear Saturation). *Define constants* $(L, M) \in R^+$ *such that* $0 < L \leq M$*. Now, define a function* $\sigma : \mathbb{R} \to \mathbb{R}$*.* σ *is said to be a* Linear Saturation *if it is continuous, nondecreasing and satisfies*

a. $s\sigma(s) > 0 \qquad \forall s \neq 0$

b. $\sigma(s) = s \qquad$ *when* $|s| \leq L$

c. $|\sigma(s)| \leq M \qquad \forall s \in \mathbb{R}$

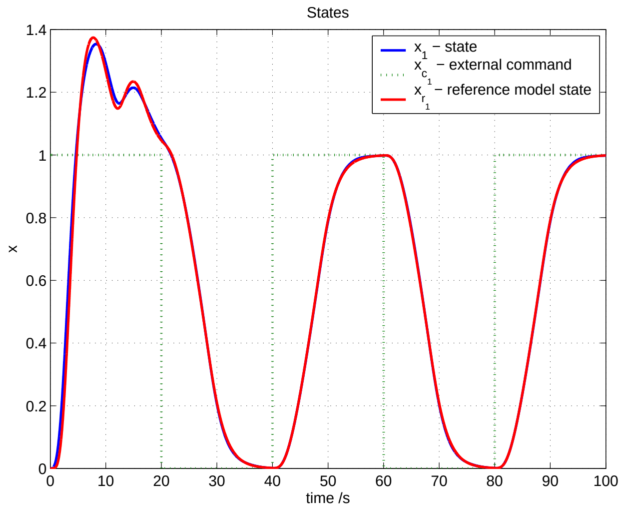

Fig. 6 State x_1 response

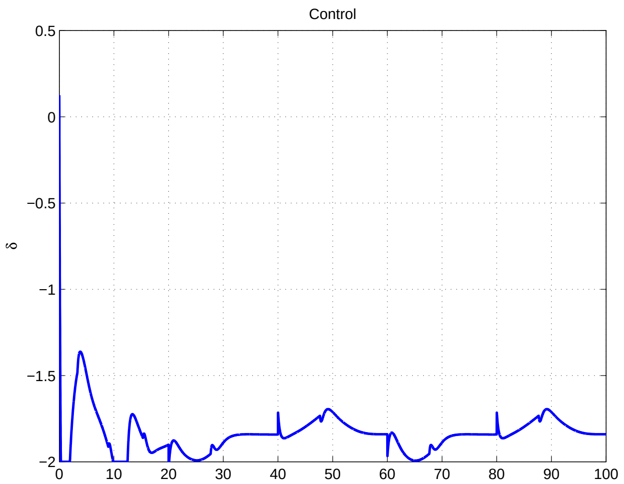

Fig. 7 Control signal

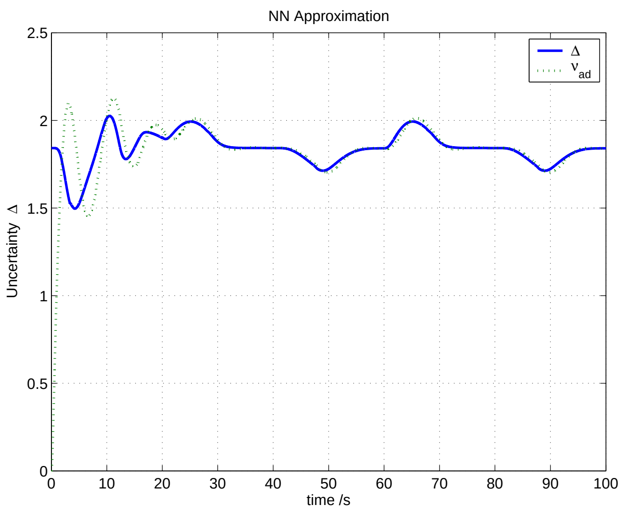

Fig. 8 Uncertainty and neural network approximation of it

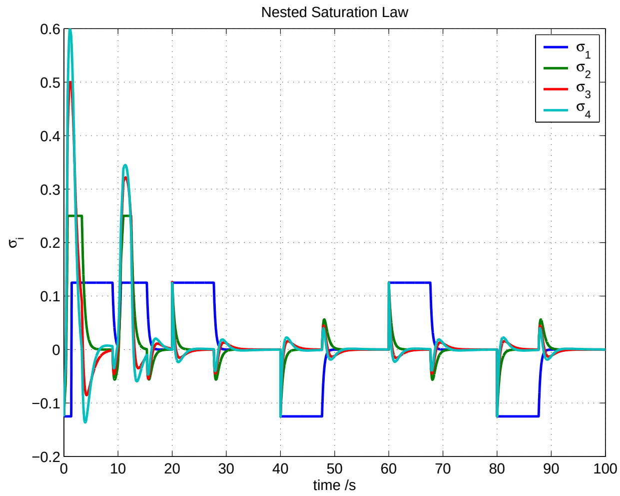

Fig. 9 Nested saturation law, saturation time history, σ_i, for $i = 1..4$

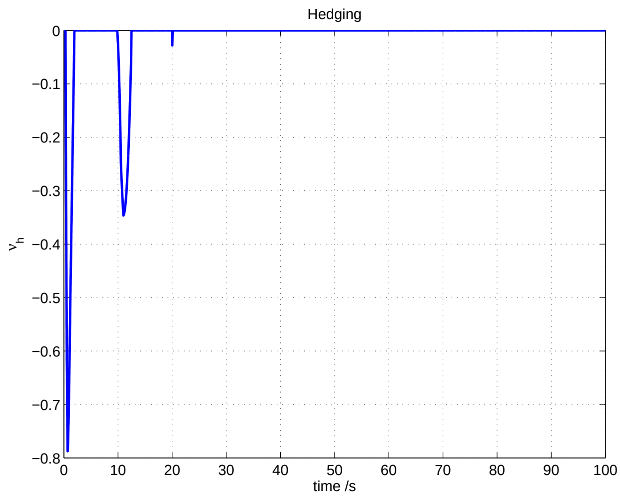

Fig. 10 Hedging signal time history, ν_h

Definition 2 ($\mathcal{L}_\infty$ norm). *For a measurable function $u : [0, \infty) \to R^m$,*

$$||u||_\infty \triangleq \sup_{t \in [0,\infty)} |u(t)|$$

Definition 3 (Asymptotic Norm[9]). *For a measurable function $u : [0, \infty) \to R^m$,*

$$||u||_a \triangleq \lim_{t \to \infty} \sup\{ \max_{1 \le i \le m} |u_i(t)| \}$$

Definition 4 (Gain Function). *A function $\gamma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is said to be a* gain function *if it is* continuous, nondecreasing *and* $\gamma(0) = 0$*. A* globally invertible gain function *is one that is a gain function that is strictly increasing and unbounded. If a gain function has the additional property of being strictly increasing then it is a class $\mathcal{K}$ function.*

Definition 5 (Asymptotic Input Output Bound[9,14]). *The system*

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

with $x \in R^n$, $u \in R^m$, $y \in R^p$ is said to satisfy an asymptotic input output bound*, with restriction $X \subset R^n$ on x_0 and restriction U on $u(\cdot)$, if there exists a gain function $\gamma_u(\cdot)$, such that for any $x_0 \in X$ and each locally essentially bounded u satisfying $||u(\cdot)||_a < U$, the trajectory $x(t)$ with initial condition $x(0) = x_0$ exists for all $t \geq 0$ and $y(t)$ satisfies*

$$||y(\cdot)||_a \leq \gamma_u(||u(\cdot)||_a)$$

Note that in contrast to input to state stability bounds, the definition of asymptotic input output bound does not prescribe a relationship between $||x(t)||$ and $||x_0||$.

Lemma 3. [9,14] *Consider the system*

$$\dot{x} = Ax + B\sigma(Kx + v) + w$$
$$y = x$$

where, $x \in R^n$. If (A,B) is stabilizable and there exists a P such that $A^T P + PA \leq 0$ and K is such that $A + BK$ is Hurwitz and

$$max\{\|v(\cdot)\|_a, \|w(\cdot)\|_a\} \leq \delta'$$

then, the above system satisfies an asymptotic input-output bound*, for all $x_0 \in R^n$ with linear gain functions $\gamma_v(\cdot)$ and $\gamma_w(\cdot)$.*

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