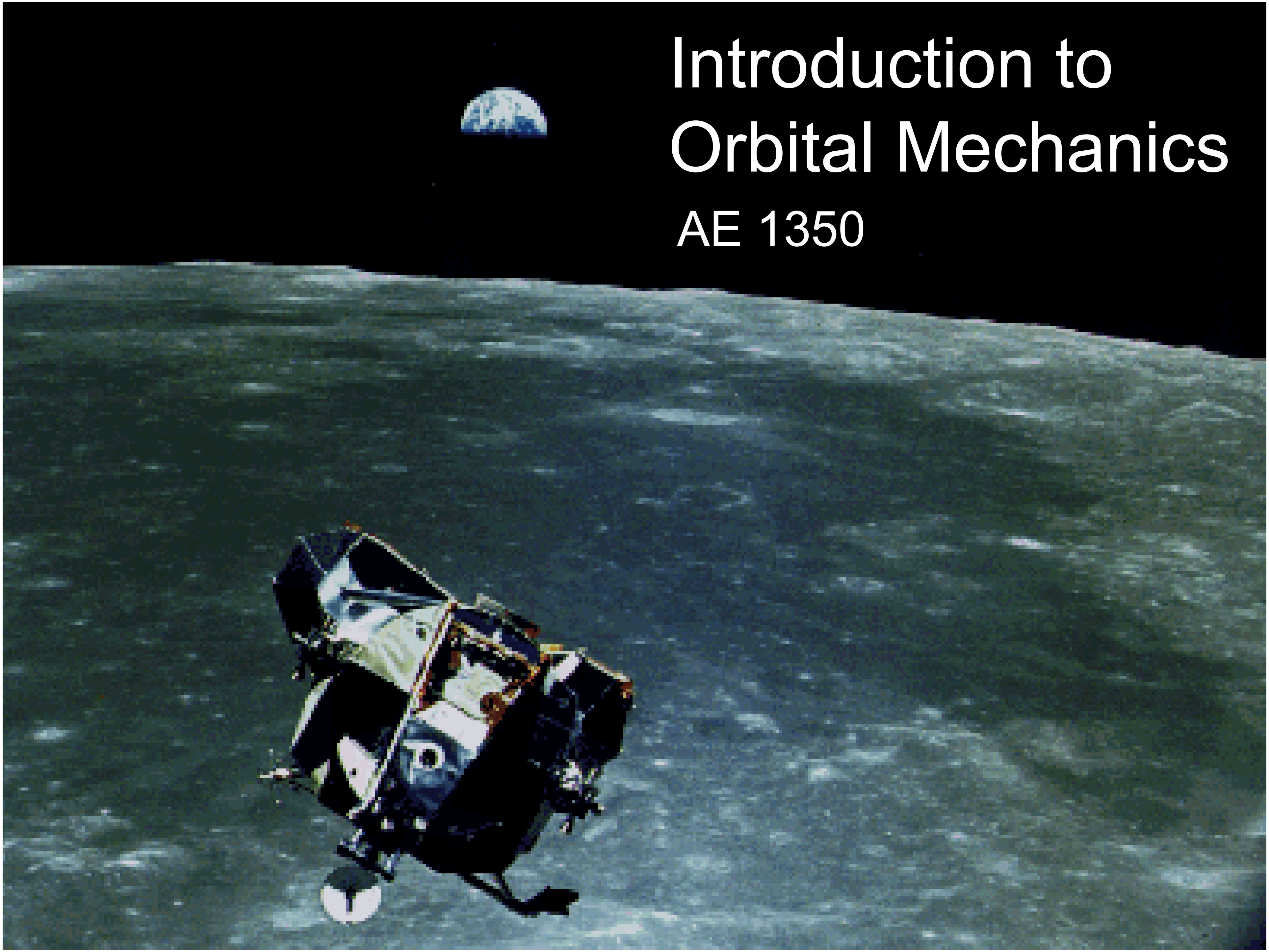
Introduction to Orbital Mechanics

AE 1350

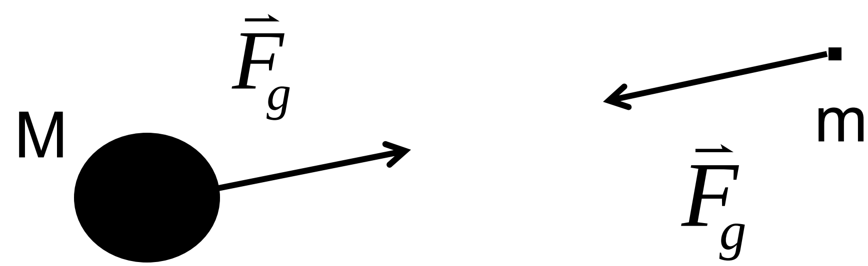


Newton's Law of Universal Gravitation

- Any two bodies attract one another with a force
  - Proportional to the product of their masses
  - Inversely proportional to the square of the distance between them

$$\boxed{\vec{\mathbf{F}}_{\mathbf{g}} = \frac{-\mathbf{GMm}}{\mathbf{r}^2}\hat{\mathbf{r}}}$$

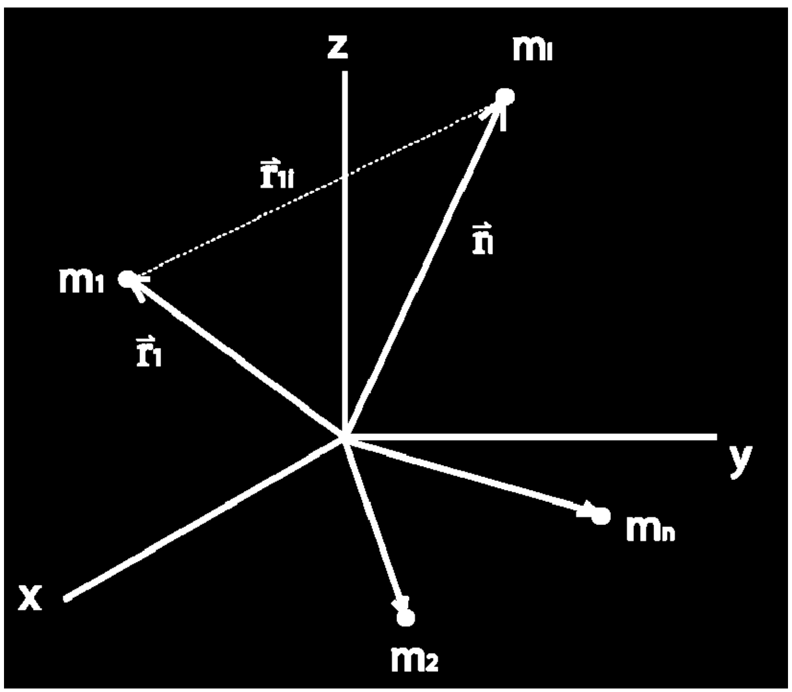
where G = 6.674 x 10^{-11} Nm^2/kg^2



This assumes a <u>spherical mass distribution</u> of the gravitational bodies

The N-Body Problem

- At any given time, a body in space is being acted upon by several gravitational forces, and other forces like drag, solar pressure, thrust, etc.
- If we consider *just* the gravitational forces, assuming each body has a spherical mass distribution:



Let's study the motion of body m_i

r_1 is the position of the body 1 relative to our origin
r_{1i} is the position of the body i relative to body 1

The N-Body Problem

- The vector sum of all such gravitational forces action on body m_i is:

$$\vec{F}_g = \sum_{\substack{j=1 \\ j \neq i}}^{n} \vec{F}_{g_j} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{-Gm_i m_j}{r_{ji}^{\,2}} \hat{r}_{ji}$$

$$\vec{F}_g = -Gm_i \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j}{r_{ji}^{\,2}} \hat{r}_{ji}$$

N-Body Equation of Motion

- Newton's 2nd Law:

$$\ddot{\vec{r}}_i = \frac{\vec{F}}{m_i} \qquad i = 1\ldots n$$

- If gravitational forces are the only forces:

$$\ddot{\vec{r}}_i = \frac{\vec{F}_g}{m_i} = -G\sum_{\substack{j=1 \\ i\neq j}}^{n} \frac{m_j}{r_{ji}^{2}}\hat{r}_{ji} \qquad i = 1\ldots n$$

- This is a 2nd order, nonlinear vector differential equation that does *not have* a general closed-form solution, normally solved via numerical integration

Relative Motion of 2 Bodies

Body 1: Earth

Body 2: Spacecraft

Body 3: Moon

Body 4: Sun

Body 5: Mars

Body 6: …

$$\ddot{\vec{r}}_1 = -G\sum_{j=2}^{n}\frac{m_j}{r_{j1}^2}\hat{r}_{j1}$$

$$\ddot{\vec{r}}_2 = -G\sum_{\substack{j=1\\ j\neq 2}}^{n}\frac{m_j}{r_{j2}^2}\hat{r}_{j2}$$

As we defined earlier:

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$$

$$\ddot{\vec{r}}_{12} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1$$

where $\vec{r}_{12}$ is the position of body 2 relative to body 1
and $\ddot{\vec{r}}_{12}$ is the acceleration of body 2 relative to body 1

$$\ddot{\vec{r}}_{12} = -G\left[\frac{m_1}{r_{12}^2}\hat{r}_{12} + \sum_{j=3}^{n}\frac{m_j}{r_{j2}^2}\hat{r}_{j2} - \frac{m_2}{r_{21}^2}\overset{-\hat{r}_{12}}{\cancel{\hat{r}_{21}}} - \sum_{j=3}^{n}\frac{m_j}{r_{j1}^2}\hat{r}_{j1}\right]$$

$$\ddot{\vec{r}}_{12} = \frac{-G\left(m_1 + m_2\right)}{r_{12}^2}\hat{r}_{12} - G\sum_{j=3}^{n}m_j\left(\frac{\hat{r}_{j2}}{r_{j2}^2} - \frac{\hat{r}_{j1}}{r_{j1}^2}\right)$$

Relative Motion of 2 Bodies

This be used to describe the acceleration (motion) of the spacecraft relative to the Earth, the 2nd term on the RHS accounts for the "perturbing effects" of other bodies

$$\ddot{\underline{r}}_{12} = \overbrace{\frac{-G(m_1 + m_2)}{r_{12}^{\;2}}}^{(1)} \hat{r}_{12} \overbrace{- G \sum_{j=3}^{n} m_j \left(\underbrace{\frac{\hat{r}_{j2}}{r_{j2}^{\;2}}}_{(3)} - \underbrace{\frac{\hat{r}_{j1}}{r_{j1}^{\;2}}}_{(4)} \right)}^{(2)}$$

$\ddot{\underline{r}}_{12}$ Acceleration of body 2 <u>relative</u> to body 1

(1) two body term which is a function of m_1, m_2, r_{12}
(2) gravitational acceleration caused by masses 3…n, "Perturbing Term"
(3) gravitational acceleration of body 2 due to masses 3…n
(4) gravitational acceleration of body 1 due to masses 3…n

Acceleration Contributions for a Satellite in Earth Orbit

- Acceleration contributions in G's (9.81 m/s^2) for a 200-NM Earth satellite:

Earth	0.89 (Spherical)
Earth oblateness	10^{-3}
Sun	6×10^{-4}
Moon	3×10^{-6}
Jupiter	3×10^{-8}
Venus	2×10^{-8}
Saturn	2×10^{-9}
Mars	7×10^{-10}

In this table, relative acceleration is defined as: $$\frac{-Gm_j}{r_j^{\,2}}\hat{r}_j$$

which is equivalent to the term used: $$\frac{-Gm_j\hat{r}_{j_2}}{r_{j_2}^{\,2}}$$

Relative Motion of 2 Bodies

- When we simplify to only 2 bodies:
  - $M = m_1$ (example: mass of the Earth)
  - $m = m_2$ (example: spacecraft)
  - $r = r_{12}$

$$\ddot{\vec{r}} = -\frac{G(M+m)\hat{r}}{r^2}$$

- *This* 2nd order vector differential equation has a closed form solution!

Relative Motion of 2 Bodies

- For most 2-body problems, we have a spacecraft orbiting a much more massive planetary body: m<<M

$$G(m+M) \approx GM = \mu \qquad \mu \text{ is in units of } \left(\frac{km^3}{s^2}\right)$$

- This allows us to write the more compact:

$$\boxed{\ddot{\vec{r}} = \frac{-\mu}{r^2}\hat{r}}$$

Total Energy is Conserved

- Kinetic energy: $KE = 1/2mv^2$
- Potential energy:

$$\Delta PE = -\int_{x_i}^{x_f} F(x)dx$$

Work done equals the decrease in PE

$$\Delta PE = -\int_{ref}^{r} \vec{F}_g dr = -\int_{ref}^{r} \frac{-GMm}{r^2}\hat{r}dr$$

$$\Delta PE = \left.\frac{-GMm}{r}\right|_{ref}^{r}$$

$$PE\Big|_r - PE\Big|_{ref} = -\frac{-GMm}{r} + \frac{GMm}{r_{ref}}$$

Total (Specific) Energy is Conserved

- Define $PE\big|_{ref} = 0$ at $r_{ref} = \infty$ therefore,

$$PE = -\frac{GMm}{r} = \frac{-\mu m}{r}$$ (PE will always be negative)

- Total Energy <u>per unit mass:</u>

$$\varepsilon = \frac{KE + PE}{m} = \frac{v^2}{2} - \frac{\mu}{r}$$

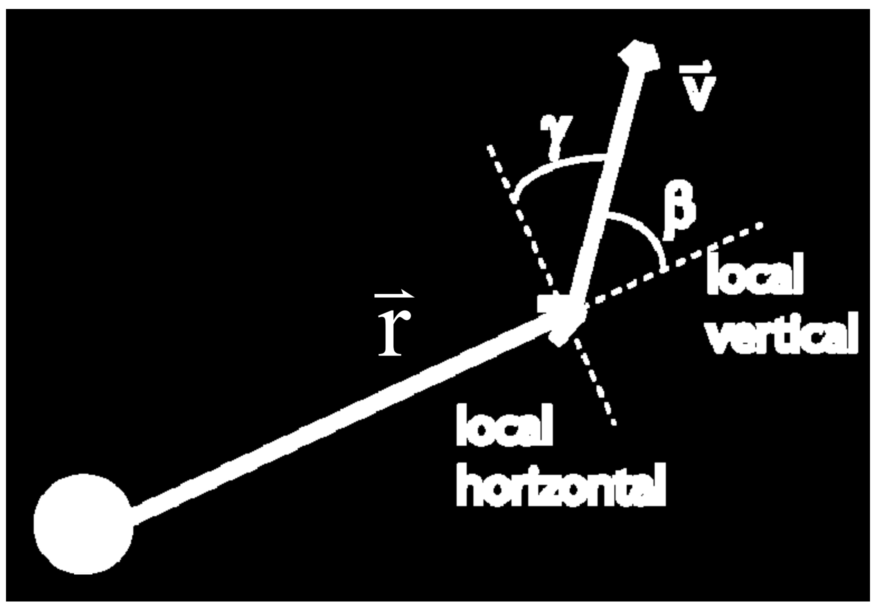
$$\boxed{\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}}$$

(Specific) Angular Momentum

- Angular momentum per unit mass:

$$\vec{h} = \vec{r} \times \vec{v}$$

- Since h is the cross-product of r and v, it must always be perpendicular to the plane containing r and v



Two-Body Motion

- Motion conserves angular momentum *and* energy
  - An object moving under the influence of a single, spherical, and constant-mass gravitational field
- Therefore:
  - Has constant angular momentum vector (h), both magnitude and direction:
    - Motion in a constant plane
    - Our 3D problem just became a 2D problem!
  - Does not lose or gain energy, but simply exchanges PE for KE and vice versa (constant ε)
    - Types of solutions characterized by ε

Two-Body Orbit Equation

- The exact analytic solution to the two-body equation of motion

$$\ddot{\vec{r}} = -\frac{\mu}{r^2}\hat{r}$$

is the *conic section* equation:

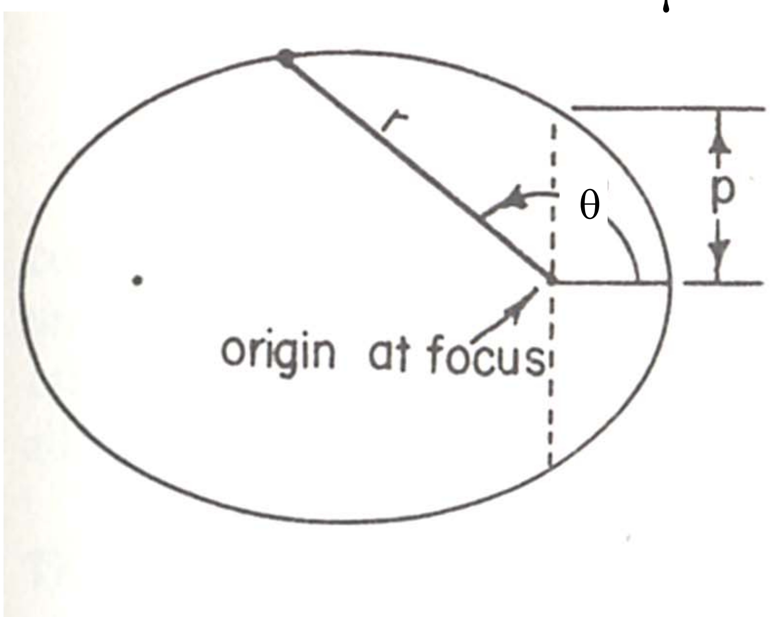
$$\boxed{r = \frac{h^2/\mu}{\left(1 + \frac{B}{\mu}\cos\theta\right)}}$$

- This is the 2-body trajectory equation, written in polar coordinates where
  - r is a function of θ and constants of motion (h, μ, B)
  - Here, θ is defined as the angle between the B and r vectors
  - B is a vector constant of integration that has yet to be defined

General Equation of a Conic Section

- This is the general equation of a conic section, written in polar coordinates with the origin located at one focus: $r = \dfrac{p}{1 + e\cos\theta}$
- So let's we define: $p = h^2/\mu$ p is termed the semi-latus rectum

  $e = B/\mu$ e is termed the eccentricity, type of conic

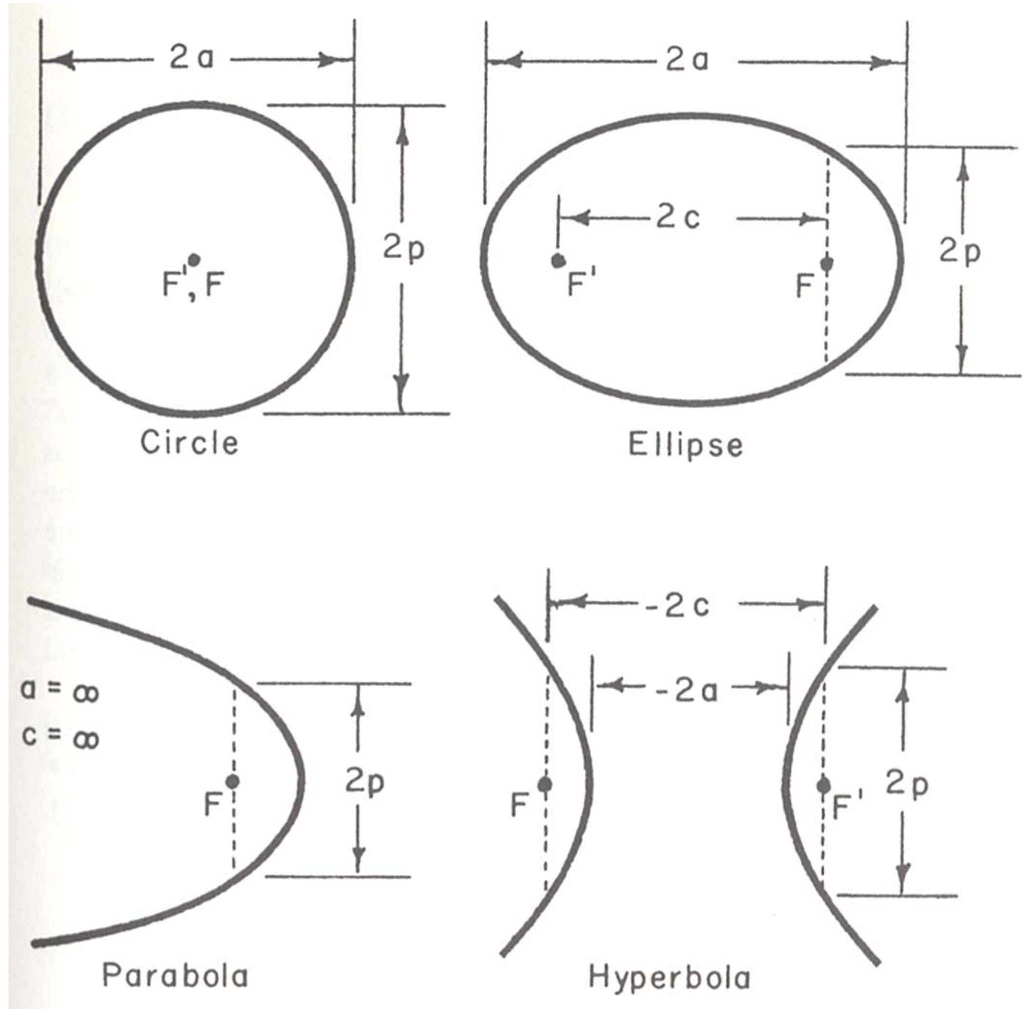


$e = 0$	Circle
$0 < e < 1$	Ellipse
$e = 1$	Parabola
$e > 1$	Hyperbola

Conic Sections

- The *prime focus*, F, marks the location of the gravitational body
- 2p is the width of each curve at the focus (latus rectum)
- 2a is the length of the chord passing through the foci (major axis)
- 2c is the distance between the two foci

$$p = a\left(1 - e^2\right) \qquad e = c/a$$



Summary of Two-Body Motion

- h vector is a constant, so motion takes place in a single plane, the orbital plane, fixed in inertial space
- The family of curves called conic sections represents the only possible paths for the orbiting object
- The focus of the conic orbit must be located at the center of the gravitational mass
- We know ε = constant;
  PE < 0 and PE -> 0 as r -> ∞, so
  - KE (and orbiting object speed) decreases as the orbiting object gains altitude
  - Speed increases as the orbiting object loses altitude

Apses

- The extreme points along a conic's major axis are referred to as "apses"
  - *periapsis*, $\theta = 0^o$ (closest point), r_p
  - *apoapsis*, $\theta = 180^o$ (farthest point), r_a

$$r_p = \frac{p}{1+e} = a(1-e)$$

$$r_a = \frac{p}{1-e} = a(1+e)$$

- For Earth, these would be the *perigee* and *apogee*

Angular Momentum and Energy

- At r_p and r_a (since velocity and radius are at right angles here)

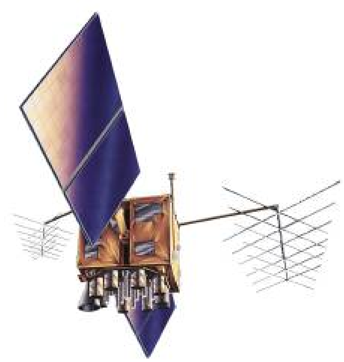
$$h = r_p v_p = r_a v_a$$

- At periapsis, the energy can be calculated as:

$$\varepsilon = \frac{v_p^{\ 2}}{2} - \frac{\mu}{r_p}$$ where: $$h^2 = p\mu = a\mu\left(1 - e^2\right)$$

$$\varepsilon = \frac{h^2}{2r_p^{\ 2}} - \frac{2r_p\mu}{2r_p^{\ 2}} = \frac{h^2 - 2r_p\mu}{2r_p^{\ 2}}$$

$$\varepsilon = \frac{h^2 - 2\mu\left(a[1-e]\right)}{2\left(a[1-e]\right)^2}$$



Energy (continued)

So:

$$\varepsilon = \frac{a\mu(1+e)(1-e) - 2a\mu(1-e)}{2a^2(1-e)^2}$$

$$\varepsilon = \frac{\mu(1+e) - 2\mu}{2a(1-e)}$$

$$\varepsilon = \frac{\mu e - \mu}{2a(1-e)}$$

$$\varepsilon = \frac{\mu(e-1)}{2a(1-e)}$$

Circle	a>0, ε<0
Ellipse	a>0, ε<0
Parabola	a=∞, ε=0
Hyperbola	a<0, ε>0

$$\varepsilon = -\frac{\mu}{2a}$$ Constant, Valid for all conic sections

Eccentricity

- Now, since h determines p, and ε determines a, the two together determine e

$$p = a\left(1 - e^2\right)$$

$$\frac{h^2}{\mu} = \frac{-\mu}{2\varepsilon}\left(1 - e^2\right)$$

$$\frac{2\varepsilon h^2}{\mu^2} = e^2 - 1$$

$$\frac{2\varepsilon h^2}{\mu^2} = e^2 - 1$$

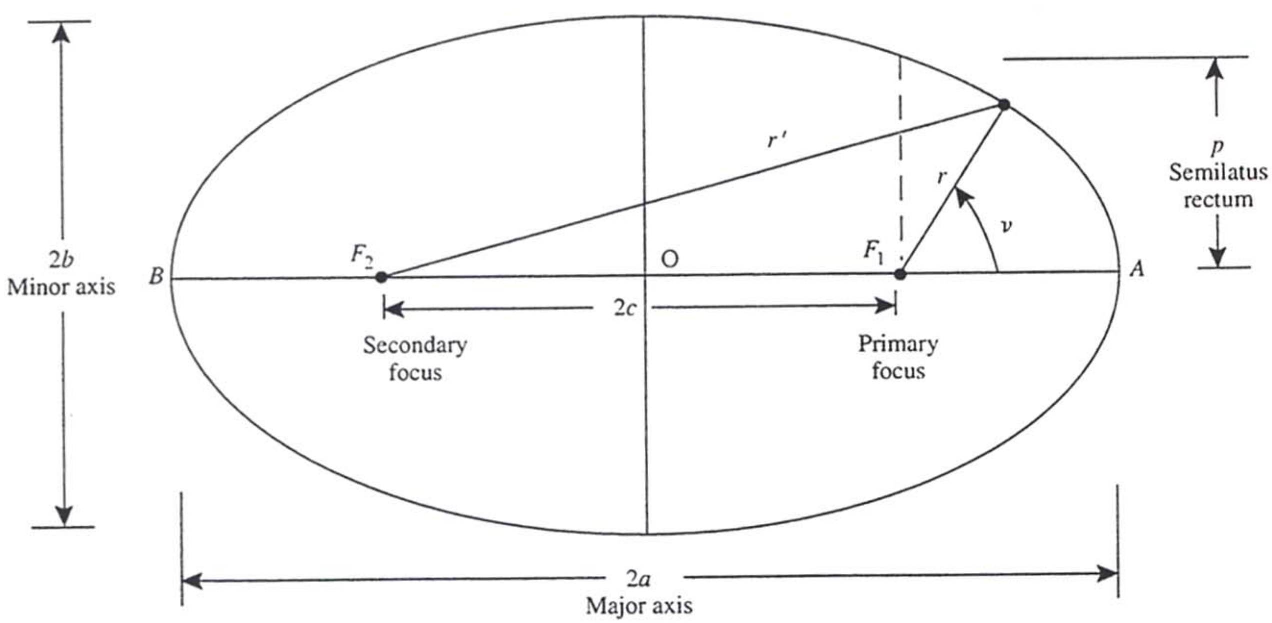
$$e = \left[1 + \frac{2\varepsilon h^2}{\mu^2}\right]^{1/2} \quad = 1 \text{ when } \varepsilon = 0$$

Conic Section Parameters

	e	a	ε	
Circle	0	> 0	< 0	Dominated by PE
Ellipse	$0<e<1$	> 0	< 0	
Parabola	1	∞	= 0	PE = KE $V_\infty = 0$
Hyperbola	> 1	< 0	> 0	KE>PE $V_\infty > 0$

Elliptical Orbits

- Planets in our solar system, most spacecraft
- An ellipse can be defined geometrically as the locus of points transcribed by a piece of string anchored at the two foci and traced out for all θ

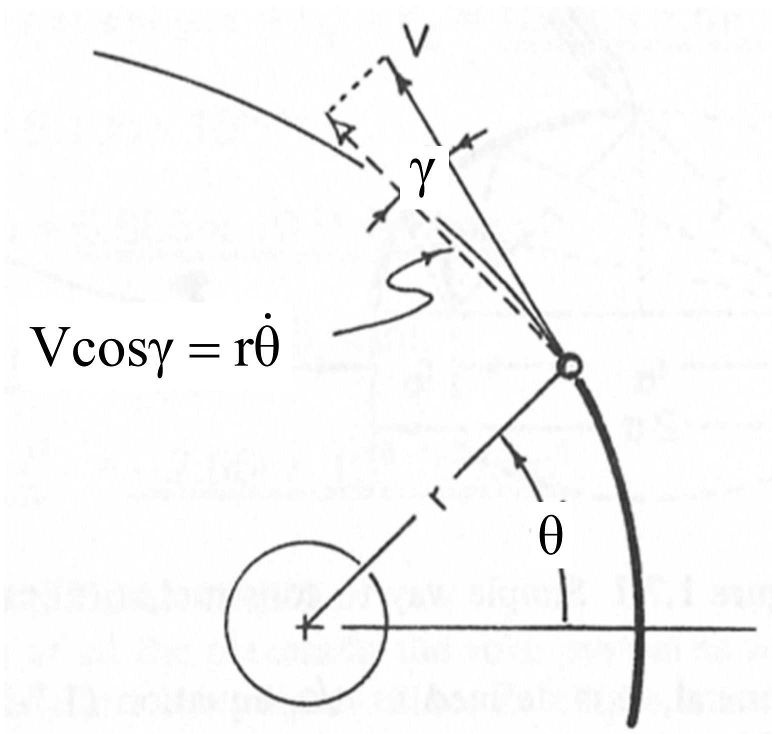


For $\theta = 0^o$ or 180^o,
$r + r' = 2a$

By definition
$r_p + r_a = 2a$

Orbital Period

- Orbital Period is defined as the time required for the spacecraft to travel once around its orbit (P)
  - Tangential velocity component, $V\cos\gamma = r\dot{\theta}$
  - Use the fact that, $h = rv\cos\gamma$



$$h = r^2\dot{\theta}$$

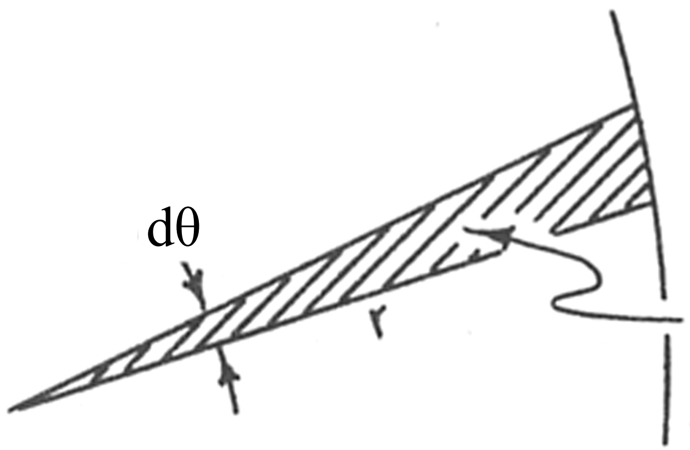
$$h = r^2\frac{d\theta}{dt}$$

$$dt = \frac{r^2}{h}d\theta$$

So: $d\theta = \frac{h}{r^2}dt$

Orbital Period

From geometry, we see that:



$$dA = \frac{1}{2} r^2 d\theta$$

$$d\theta = \frac{2}{r^2} dA$$

So,

$$dt = \frac{2}{h} dA$$

This is a mathematical statement of Kepler's 2nd law (note: h = constant)
(Planet orbits sweep out equal area per unit time as they orbit)

Over the course of one full orbit,

$$\int_0^p dt = \frac{2}{h} \int dA = \frac{2}{h} \pi ab$$

$$P = \frac{2ab\pi}{h}$$

Orbital Period

- Using the fact that: $h = \sqrt{p\mu}$
- So, $P = \dfrac{2a\pi\sqrt{ap}}{\sqrt{p\mu}}$

$$\boxed{P = \frac{2a^{3/2}\pi}{\sqrt{\mu}}}$$

- This is a mathematical statement of Kepler's 3rd law, proving that the period of an elliptical orbit depends only on the size of the semi-major axis (a is the average of r_p and r_a, that is the mean distance)

Circular Orbits

- A circular orbit is a special case of an elliptical orbit where e = 0 and r = constant = p = a,

$$P = \frac{2r^{3/2}\pi}{\sqrt{\mu}} \qquad \text{since } r = a$$

- From the energy equation, we see that:

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{-\mu}{2a}$$

- For a circular orbit where r = a, this becomes:

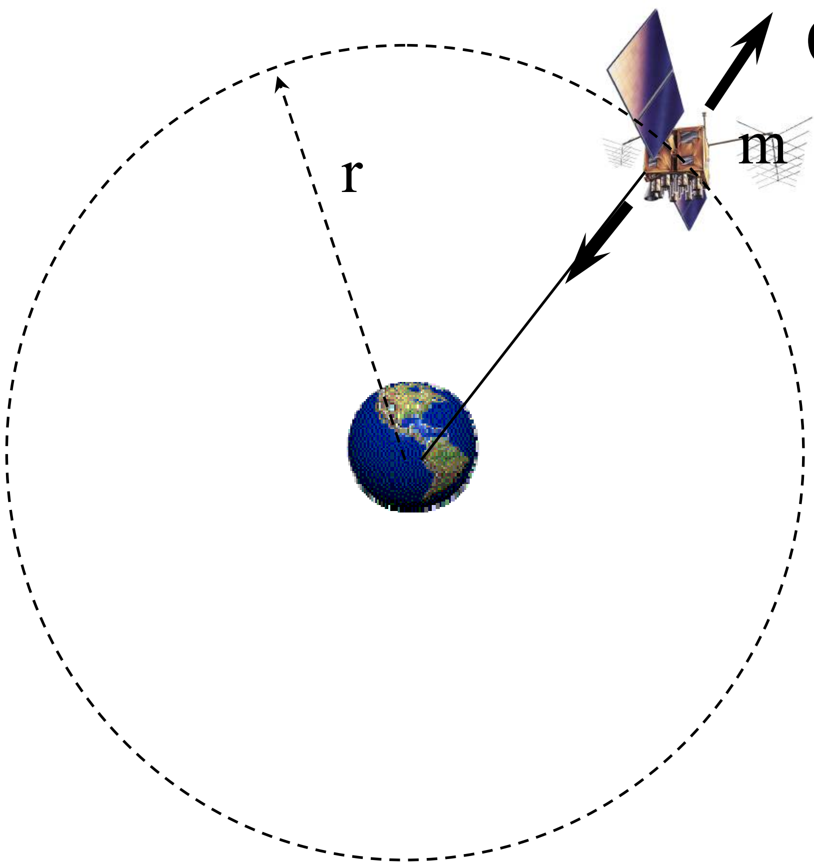
$$\frac{v^2}{2} = \frac{-\mu}{2r} + \frac{2\mu}{2r} = \frac{\mu}{2r}$$

and

$$V = \left(\frac{\mu}{r}\right)^{1/2}$$

Note that V decreases as r increases and V increases as r decreases

Circular Orbits



Centrifugal force = mV^2/r

Force due to gravity = GMm/r^2

$$GMm/r^2 = mV^2/r$$

$$GM/r = V^2$$

$$V = \left(\frac{\mu}{r}\right)^{1/2}$$

Low Earth Orbit (LEO)

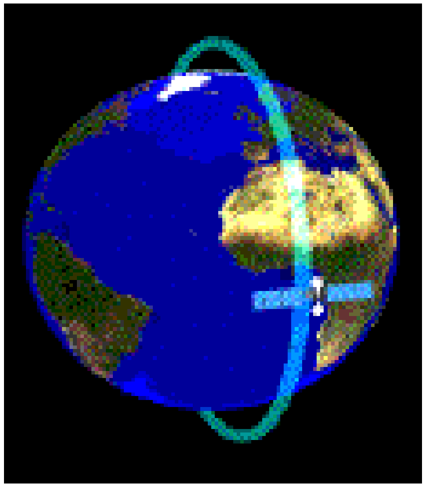
- Defined as all orbits below geostationary
- Lowest energy required to achieve
- Almost all LEO missions between
  - The atmosphere ~400,000 ft
    (so the orbit will last a while…)
  - The Van-Allen Belts ~400 nm
    (radiation trapped in the magnetic field of the Earth, a dangerous place to be)

Low inclination LEO

- *Inclination of orbit = launch latitude* is the minimum required-energy orbit
- So inclination of most US manned missions has been approximately the latitude of Kennedy Space Center (KSC), about 28 degrees, with launches almost due East
- The lower the latitude, the lower the required energy due to the spin of the Earth
  – Cheaper to launch from Florida than North Carolina…

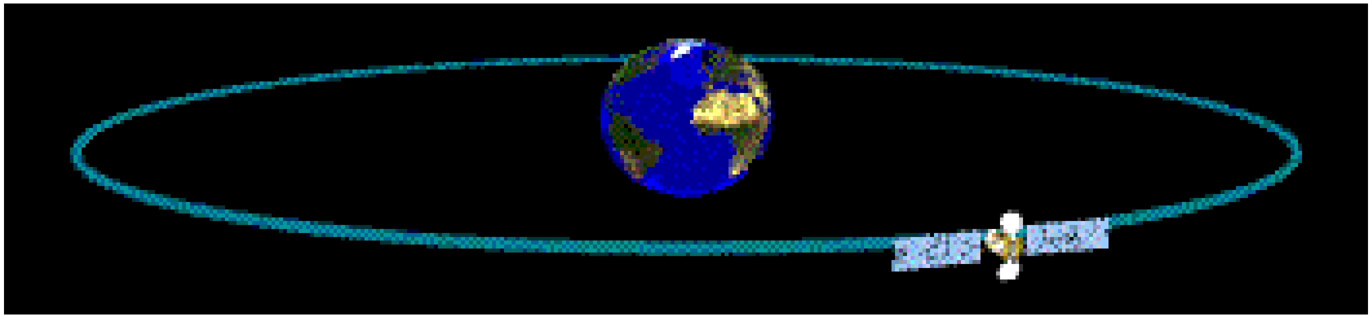
Polar LEO

- Polar orbits have an inclination near 90 degrees
- Have the advantage that they pass over the entire planet at regular intervals
  - Which has many uses…
- Takes more energy to get there than for low inclination orbits

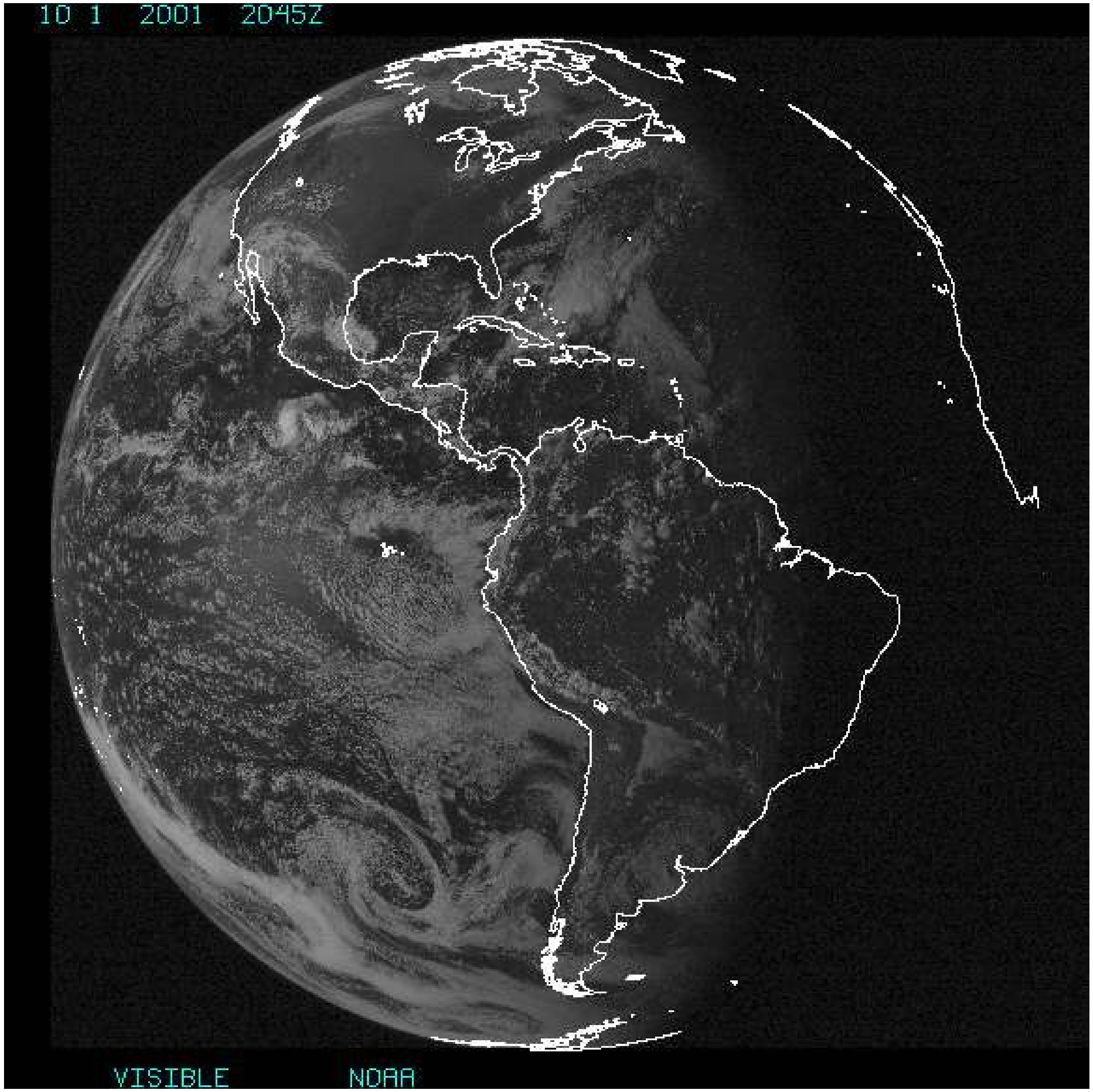


Geostationary Orbit (GEO)

- Circular
- Equatorial (inclination is zero)
- Radius is such that angular rate of orbit is the same as the angular rate of the Earth
- Radius is about 23,000 nm, period 23.934 hours
- Because it's only one radius, and one inclination – it's getting very crowded up there!

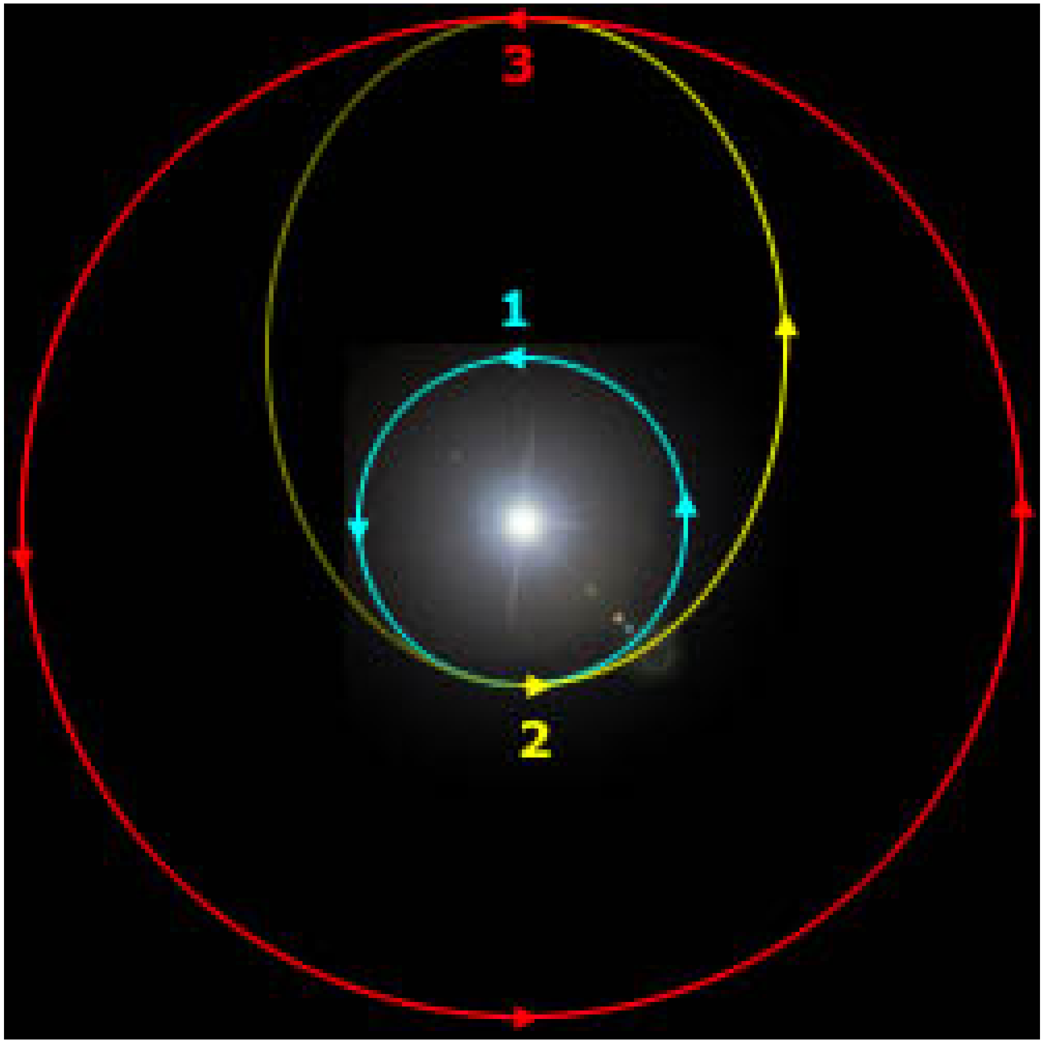


GOES-8 Typical Image



Orbital Transfers: Elliptical Orbits

- Hohmann transfer requires minimum delta-V



Escape Speed, Parabolic Orbit

- *Escape Speed* is velocity required such that the spacecraft will reach ∞ at 0 speed (will never return)
- At ∞, $\varepsilon = 0$ => $-\mu/(2a) = 0$, $a = \infty$ => parabolic orbit
- So, to calculate the velocity required to escape a body's gravitational force

$$\frac{v_{esc}^{\ 2}}{2} = -\frac{\mu}{r} = 0$$

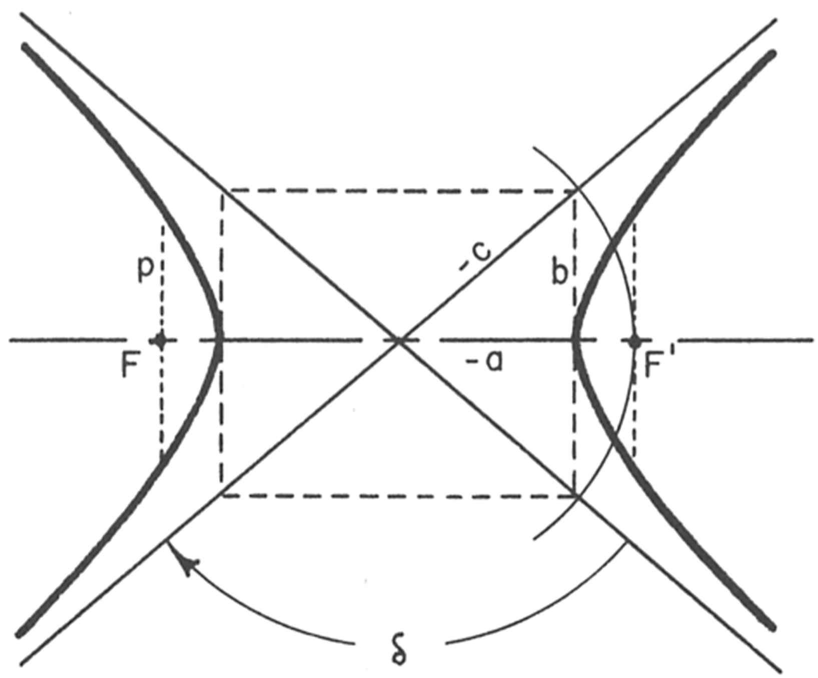
$$v_{esc} = \sqrt{\frac{2\mu}{r}}$$

Independent of direction!

- Note that: $v_{esc} = \sqrt{2}v_{circ}$ and that as r increases, v_{esc} decreases
- Orbital speed is "halfway to anywhere"

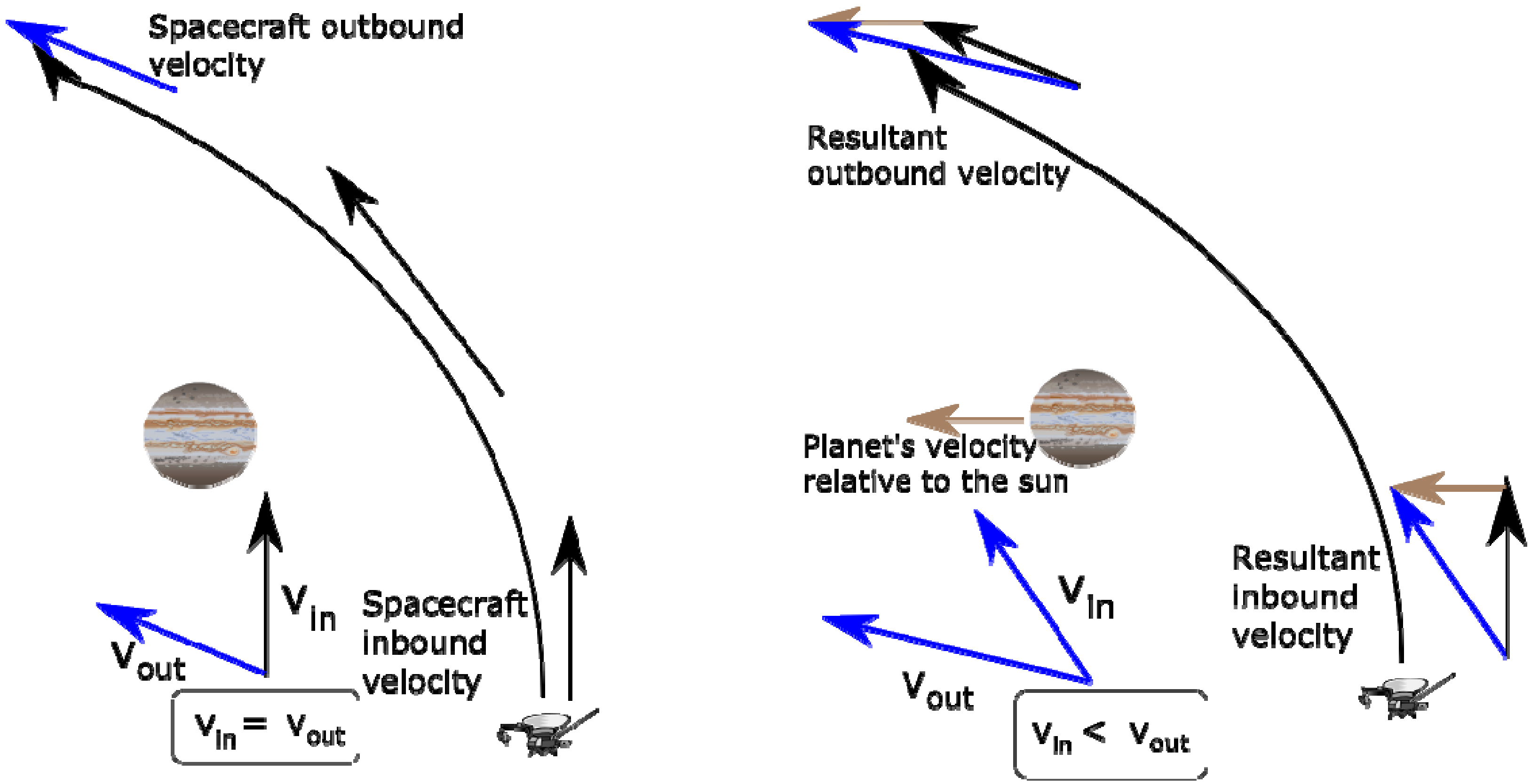
Hyperbolic Orbits

- A hyperbolic path is required when we define an orbit that reaches infinity with some nonzero velocity



Gravitational Slingshot, Hyperbolic Orbit

- Use of gravity of an object to alter the path and/or speed of a spacecraft



(wikipedia)

Lagrangian Points

- For the *restricted* *<u>three</u>-body problem*
  - Planar motion of a spacecraft relative to two other masses (which orbit each other)
    e.g., motion of spacecraft with respect to the Earth/Sun or Earth/Moon
- 5 equilibrium points for the spacecraft exist

Lagrangian Points