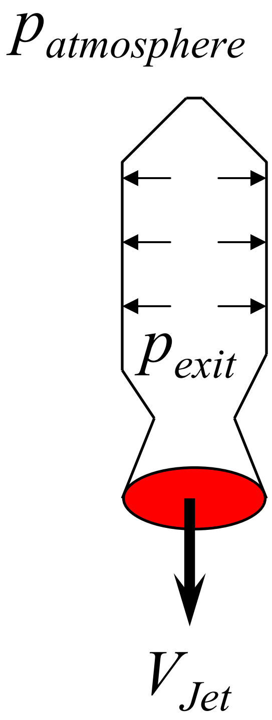
Introduction to Rocketry

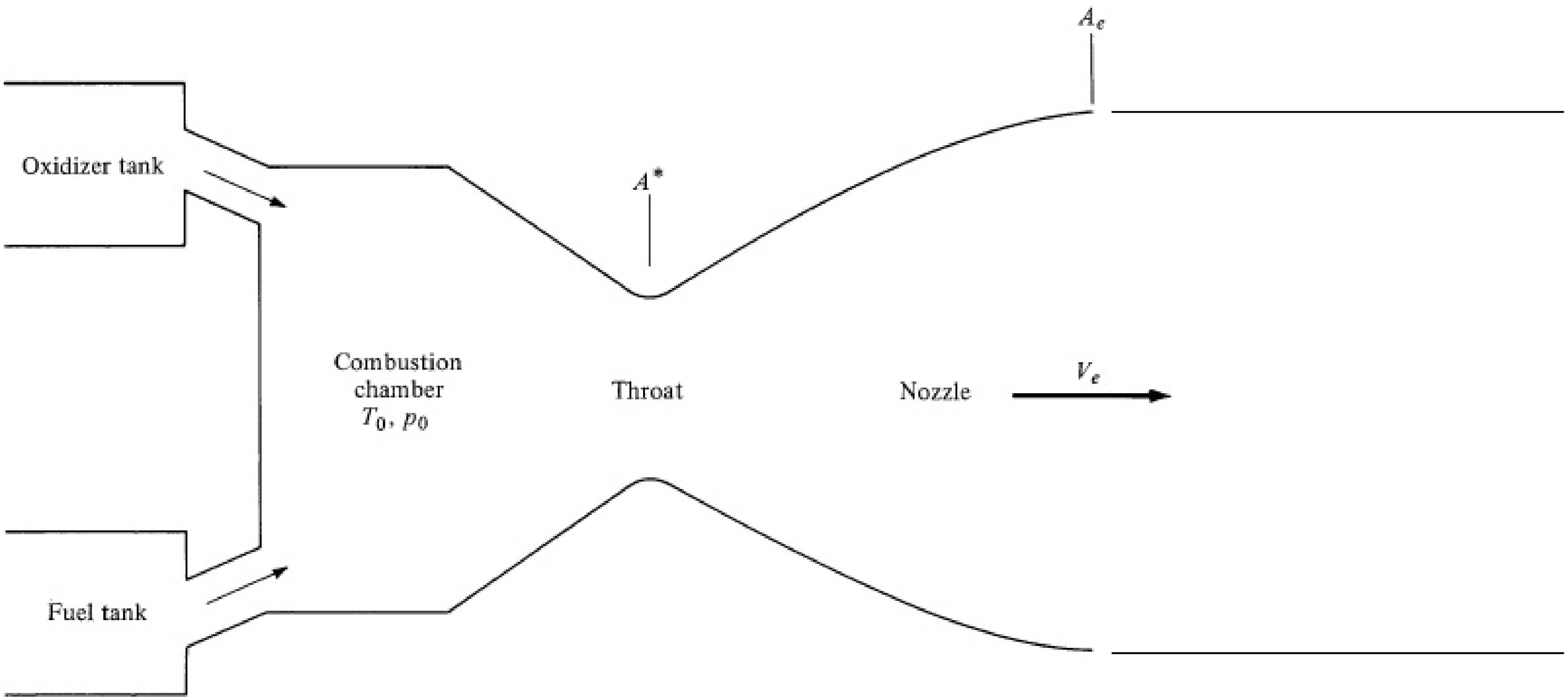
AE 1350

Rocket Motor Thrust

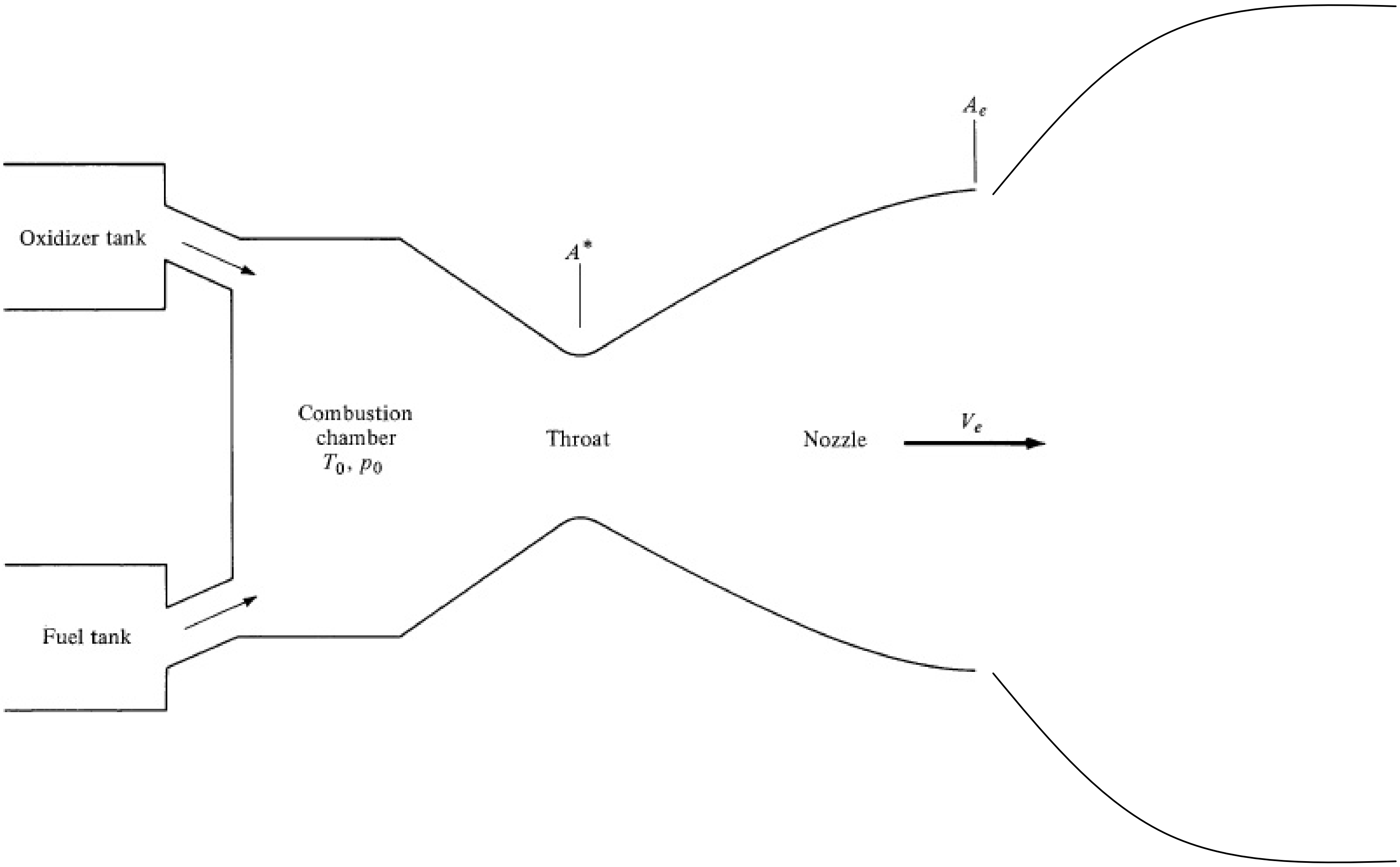


- Thrust depends on two factors:
  - Rate at which momentum leaves the rocket through the nozzle
  - Exit pressure p_{exit} and exit area A_{exit}
- $T = -dm/dt\ V_{jet} + (p_{exit} - p_{atmosphere})\ A_{exit}$
- For well designed rockets:
  - $p_{exit} = p_{atmosphere}$
  - $T = -dm/dt\ V_{jet}$
- Notice the negative sign:
  - the mass m of the rocket decreases, dm/dt is thus a negative quantity

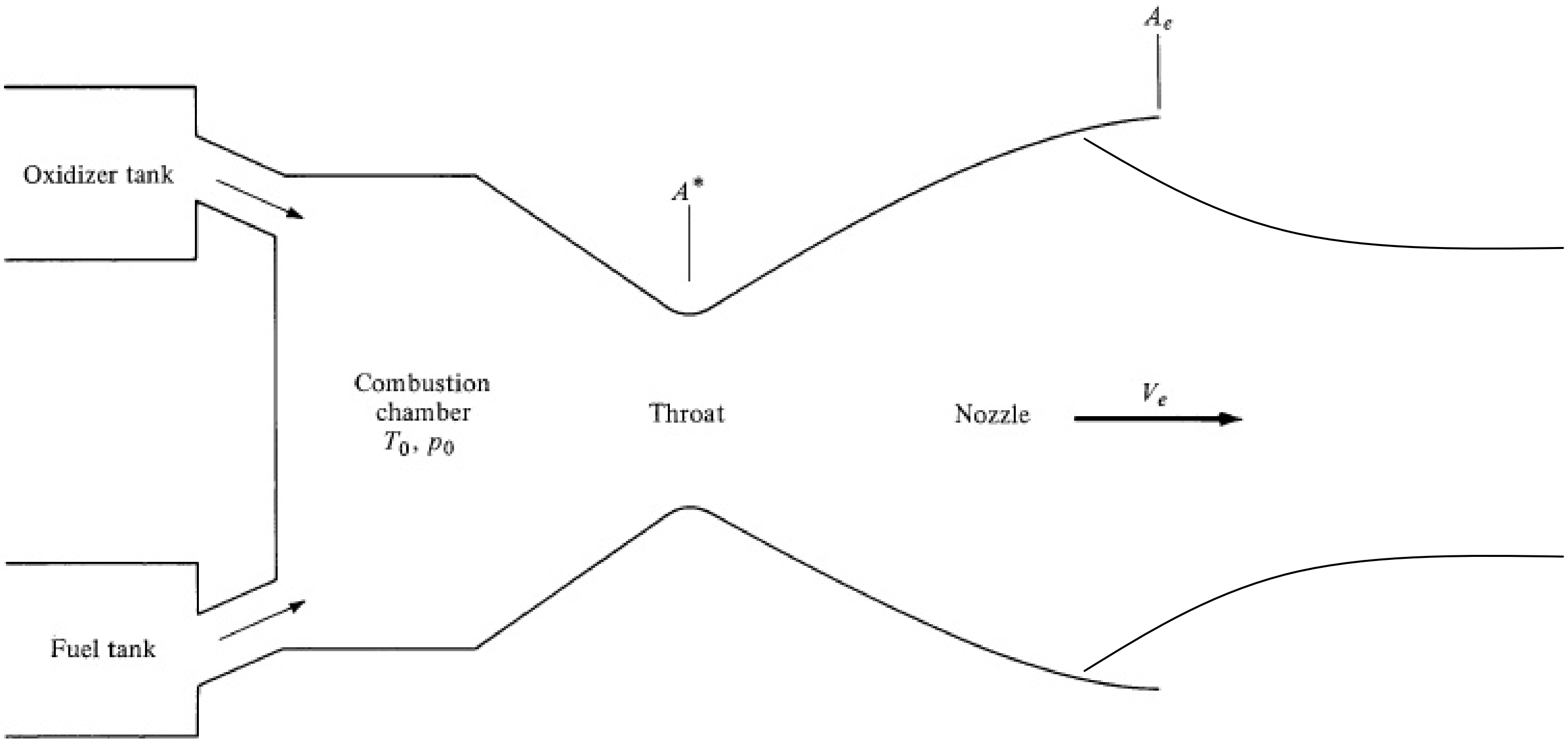
Nozzle Expansion (Ideal)



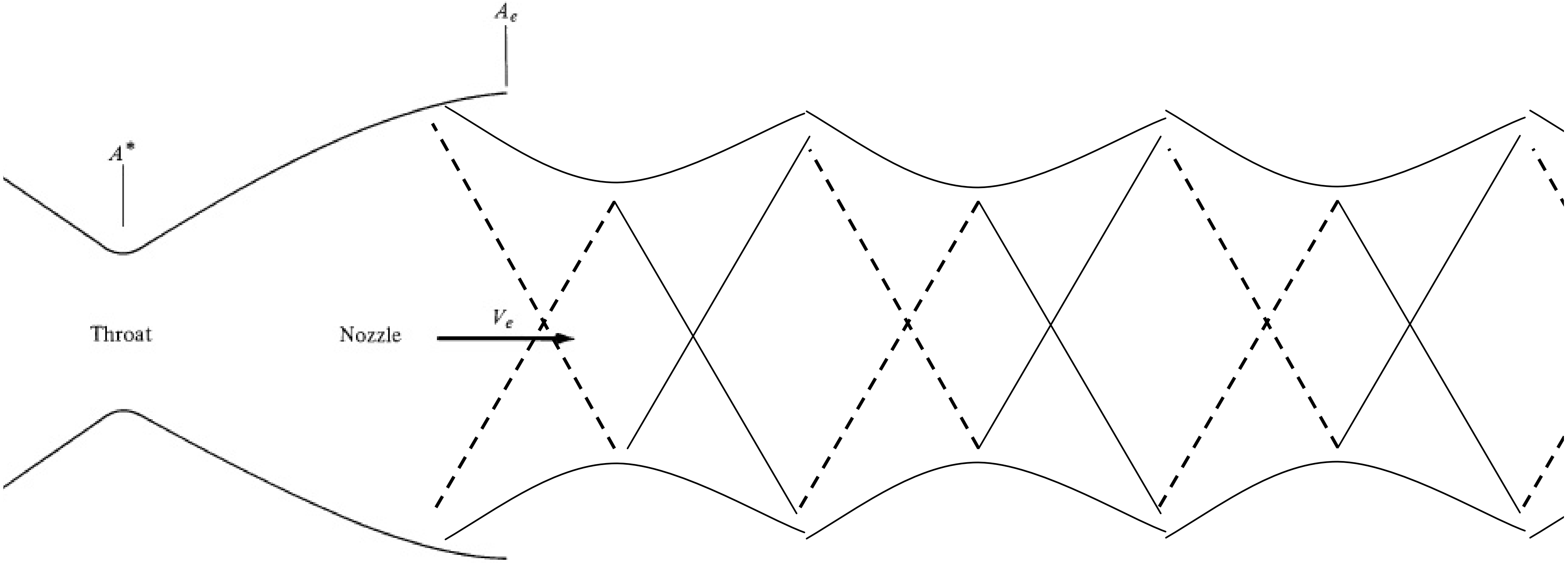
Nozzle Expansion (Under Expanded)



Nozzle Expansion (Over Expanded)



Nozzle Expansion
(Over Expanded, Supersonic Exit Velocity)



Shock Diamonds!

Nozzle Expansion (Over Expanded)

XCOR Aerospace / Mike Massee

Final Velocity of the Rocket

- From the previous slide,
  Thrust: $T = - dm/dt\ V_{jet}$
- This thrust is used to accelerate the rocket, and the payload it carries
- From Newton's second law, $F = ma$
  $m\ dV/dt = T = - dm/dt\ \ V_{jet}$
- We can write the above equation as:
  $dV = - V_{jet}\ dm/m$
- Integrate:
  $\Delta V_{rocket} = V_{jet} \ln(m_{start}/m_{end})$
  Depends only on mass ratio, and V_{jet}

How can the final velocity of the rocket be maximized?

- From the previous slide:
  Change in the speed of the rocket (and payload it carries is given by):

$$\Delta V_{rocket} = V_{jet} \ln(m_{start}/m_{end})$$

- We must increase mass of the rocket at the start by loading it up with fuel
- We must minimize the mass of all other components
  - Structure
  - Motors
  - Avionics
  - What’s left is payload

Specific Impulse

- Specific Impulse is how long a pound of fuel can develop a pound of thrust
  $T = -I_{sp}\, dm/dt\, g$
- Thrust is Specific Impulse, multiplied by the mass flow rate of propellants and the acceleration of gravity
- We can relate I_{sp} and V_{jet}:
  $T = -dm/dt\ \ V_{jet} = -I_{sp}\, dm/dt\, g$
  $V_{jet} = I_{sp}\, g$
- So, either I_{sp} and V_{jet} are a good “figure of merit” for different rocket motor choices

Rocket Motor Options

V_{jet} (km/second)	I_{sp} (seconds)	Technology
1.6 – 2.6	170 – 280	Solid Rockets (e.g. Shuttle Solid Rocket Booster, 269 sec)
1.9 – 3.4	200 – 350	Hydrocarbon Liquid Fuel (eg. Kerosene, Saturn V main engine, 255 sec.)
4.4	455	Liquid Hydrogen and Liquid Oxygen (eg. Space Shuttle Main Engine: 453 sec.)
3.0 – 5.0	300 – 500	Nuclear Hydrogen
100 – ?	10,000 – ?	Ion (low thrust)

Optimizing a Rocket

- Definitions
  - m_L payload mass
  - m_p propellant mass
  - m_s structure mass
  - m_0 initial mass $= m_L + m_p + m_s$
  - m_f final mass $= m_L + m_s$

  - Payload ratio $\pi = \dfrac{m_L}{m_0}$ Structural ratio $\varepsilon = \dfrac{m_s}{m_s + m_p}$

- We have

$$\Delta V = V_{jet} \ln \frac{m_0}{m_f}$$

Optimizing a Rocket

- To simplify

$$\frac{m_f}{m_0} = \frac{m_L + m_s}{m_L + m_s + m_p} = 1 - \frac{m_p}{m_0}$$

$$= 1 - \frac{m_s + m_p}{m_0} \frac{m_p}{m_s + m_p}$$

$$= 1 - (1 - \pi)(1 - \varepsilon)$$

$$= 1 - 1 + \pi + \varepsilon - \pi\varepsilon$$

$$= \pi + (1 - \varepsilon)\pi$$

- To get change in velocity:

$$\Delta V = V_{jet} \ln\left[\frac{1}{\varepsilon - (1 - \varepsilon)\pi}\right]$$

$$= -V_{jet} \ln[\varepsilon - (1 - \varepsilon)\pi]$$

Optimal a Single-Stage Rocket

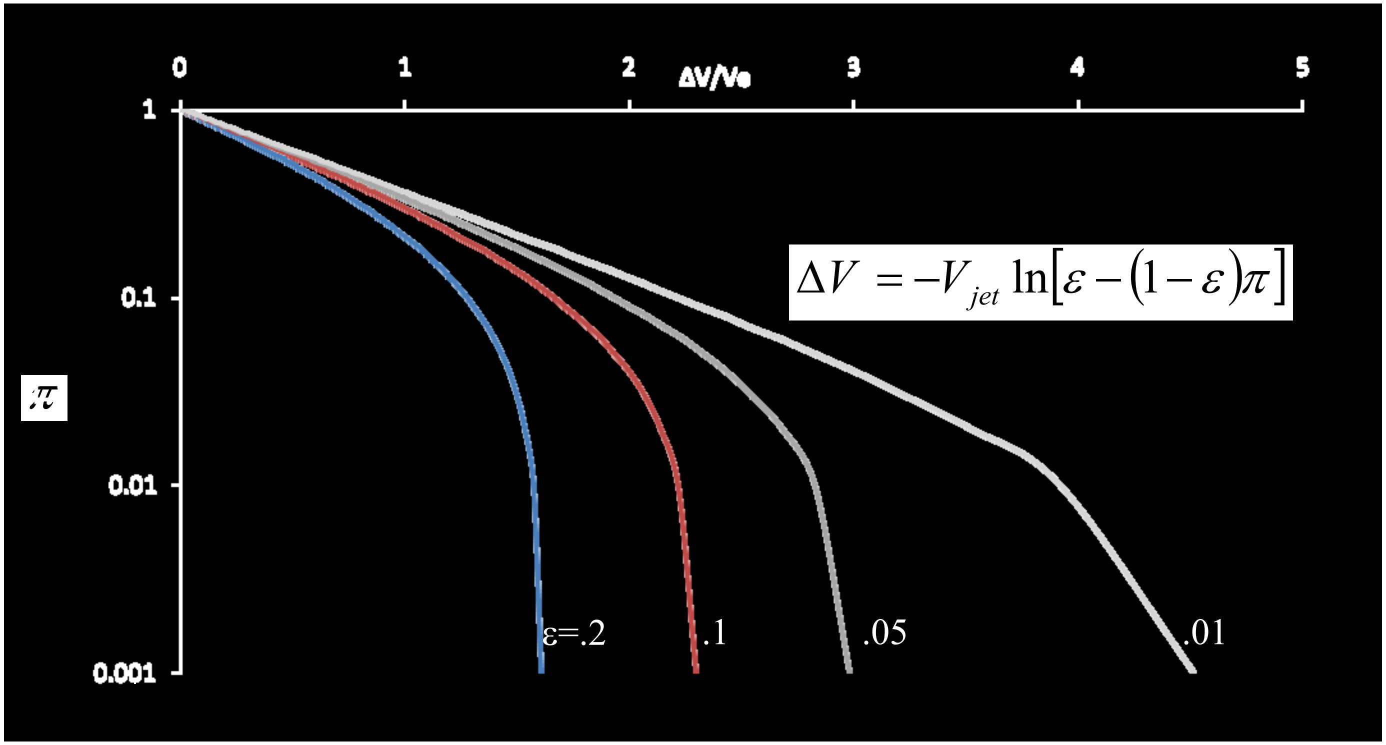
- If no payload, $\pi = 0$

  then $\Delta V = -V_{jet} \ln \varepsilon$

- Best ever is about $\varepsilon = 0.1$

  which creates a maximum practical speed with single stage

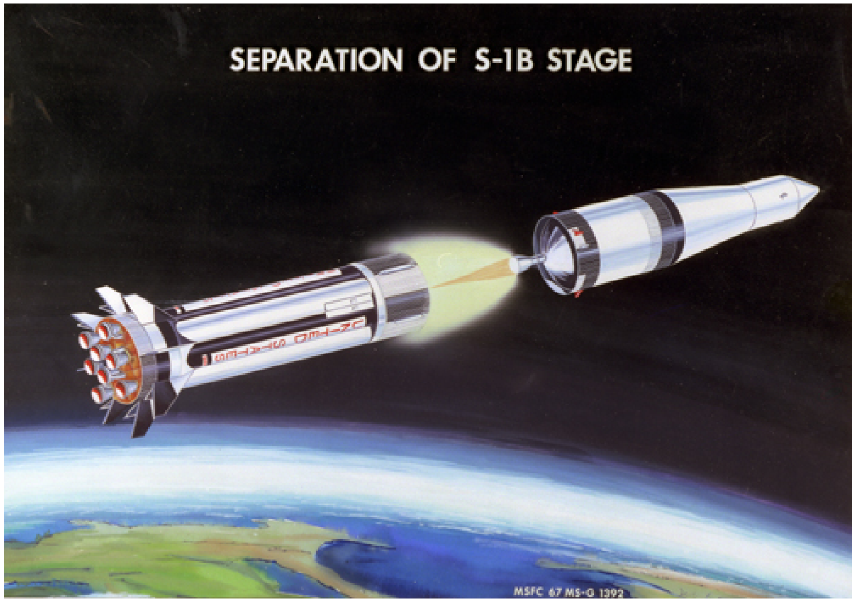
Optimal a Single-Stage Rocket

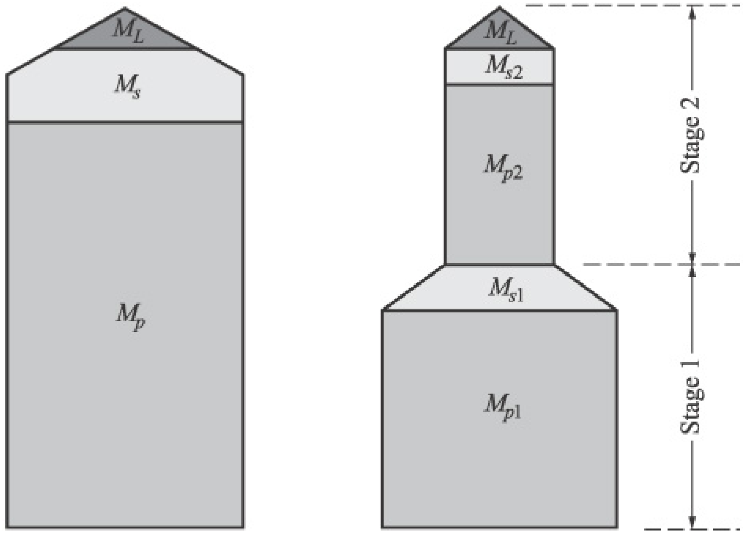


A SSTO rocket with structural ratio of 0.1 can (theoretically) achieve a ΔV of ≈ 2.3 times its exhaust velocity (no payload)

Single Stage vs. Multistage

- A multi-stage rocket discards earlier stages when able (fuel tanks burned, engines not needed)
- This decreases mass of the rocket and the end of later stages, and increases possible ΔV





Single-stage

(*a*)

Double-stage

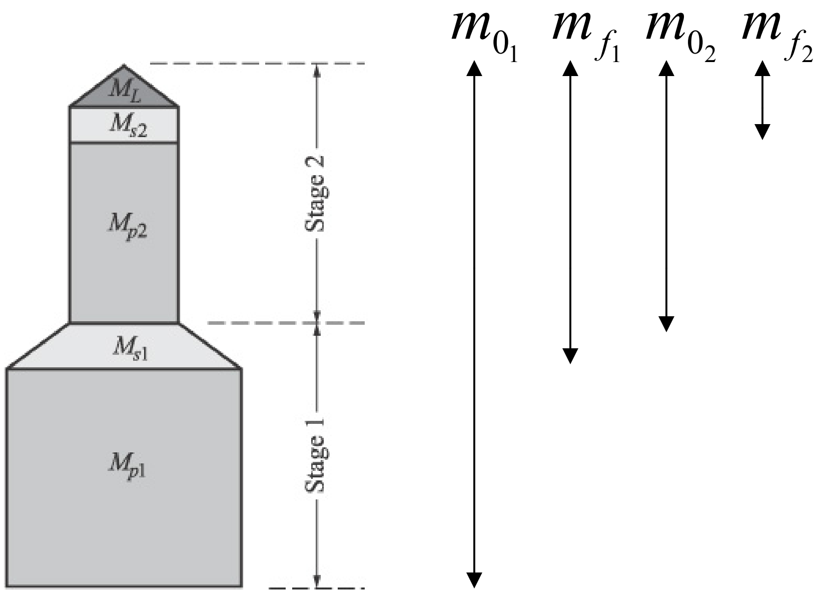
(*b*)

Optimum Multistage Rocket

- Define structural and “payload” factors
  (All future stages are payload to current stage)

$$\varepsilon_k = \frac{m_{s_k}}{m_{s_k} + m_{p_k}}, \quad \pi_k = \frac{m_{0_{k+1}}}{m_{0_k}}$$

- Overall payload factor: $\pi_* = \frac{m_L}{m_{0_1}}$



Optimum Multistage Rocket

- Total velocity change:

$$\Delta V = -\sum_{k=1}^{N} V_{jet_k} \ln\left[\varepsilon_k - (1-\varepsilon_k)\pi_k\right]$$

- Overall payload ratio is a product of individual ones:

$$\pi_* = \frac{m_L}{m_{0_1}} = \frac{m_L}{m_{0_N}} \frac{m_{0_N}}{m_{0_{N-1}}} \cdots \frac{m_{0_2}}{m_{0_1}}$$

$$= \prod_{k=1}^{N} \pi_k$$

Optimum Multistage Rocket

- Assume best values from “state of the art”: V_{jet_k}, ε_k
- Pick required ΔV
- Maximize π_* (minimize initial launch mass) by changing π_k (relative size of stages):

$$\pi_* = \prod_{k=1}^{N} \pi_k$$

$$\ln \pi_* = \sum_{k=1}^{N} \ln \pi_k$$

Optimum Multistage Rocket

- Introduce Lagrange multiplier λ
- Augment cost function:

$$\ln \pi_* = \sum_{k=1}^{N} \ln \pi_k + \underline{\lambda \left\{ \frac{\Delta V}{N} + V_{jet} \ln\left[\varepsilon + (1-\varepsilon)\pi_k\right] \right\}}$$

Constraint: $\Delta V = -\sum_{k=1}^{N} V_{jet} \ln\left[\varepsilon - (1-\varepsilon)\pi_k\right]$

- Take partial derivative and set equal to zero:

$$\frac{\partial \ln \pi_*}{\partial \pi_k} = \frac{1}{\pi_*} + \frac{\lambda V_{jet}(1-\varepsilon)}{\varepsilon + (1-\varepsilon)\pi_k} = 0$$

Optimum Multistage Rocket

- Solve for payload ratios:

$$\pi_k = \frac{-\varepsilon}{(1-\varepsilon)(1+\lambda V_{jet})}$$

- Need to eliminate Lagrange multiplier. Substitute into ΔV equation:

$$\Delta V = -\sum_{k=1}^{N} V_{jet} \ln\left[\varepsilon - \frac{\varepsilon}{1+\lambda V_{jet}}\right]$$

use to get λ

$$\lambda = \frac{e^{-\beta}}{V_{jet}\left(\varepsilon - e^{-\beta}\right)}, \quad \beta = \frac{\Delta V}{N V_{jet}}$$

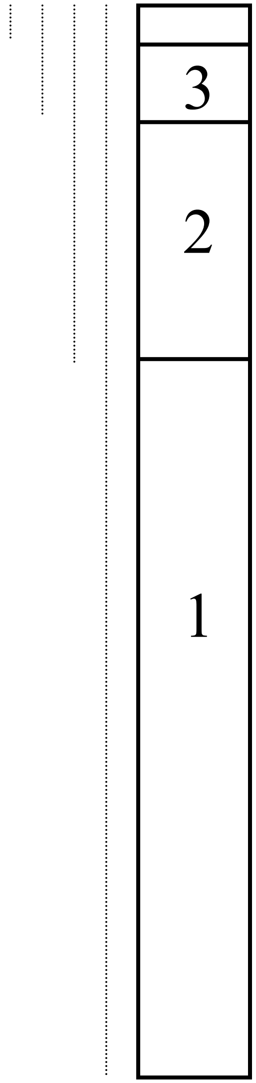
Optimum Multistage Rocket

- The final answer:

$$\pi_k = \frac{e^{-\beta}}{V_{jet}\left(\varepsilon - e^{-\beta}\right)}$$

  the same for every stage!

- Each stage smaller (in mass) than predecessor,
  e.g. $\pi = 0.33$



Saturn V

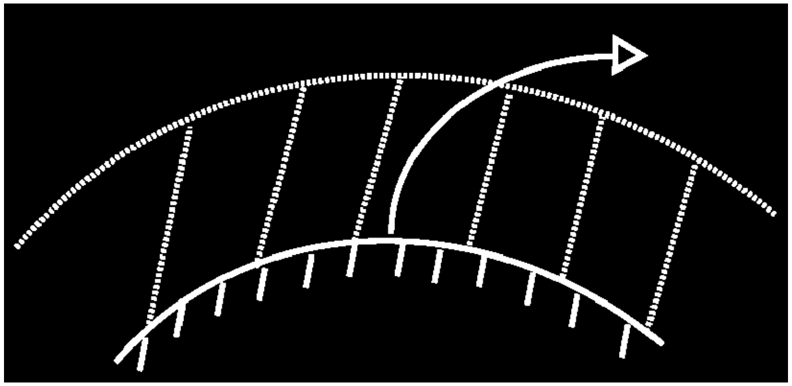
- Thrust:
  - Stage 1: 7.7M pounds
  - Stage 2: 1.1M pounds
  - Stage 3: 225k pounds

Optimum Multistage Rocket

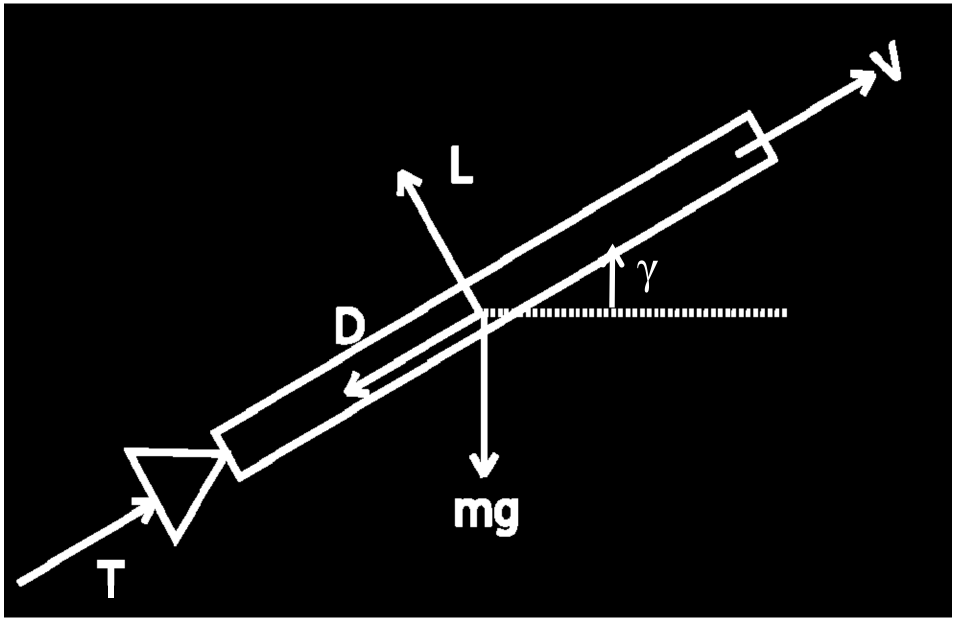
- Going beyond simplified version:
  - More ΔV for stages with higher I_{sp} (V_{jet})
  - Stages with higher I_{sp} (V_{jet}) should be above/after those with lower
- Saturn V
  - I_{sp} Kerosene/LOX: 304 sec (stage 1)
  - I_{sp} LOX/LH2: 421 sec (stages 2 and 3)
- Space Shuttle
  - I_{sp} solids: 269 sec (used 2 minutes)
  - I_{sp} main: 453 sec (used for entire launch)

Ascent Performance

- The idealized rocket equation is useful for high thrust-weight vehicles operating in space (no external forces, instantaneous ΔV, perfectly expanded nozzle flow, no gimbaling)
- These assumptions are not generally applicable to launch vehicles where:
  - Aerodynamics are significant
  - Atmospheric pressure changes
  - Flight path angle changing
  - Burn time is not small



Ascent Prediction



Assume (for now) that T lies along V, then summing forces in the velocity direction yields:

$$T - D - mg\sin\gamma = m\frac{dV}{dt}$$

Where,

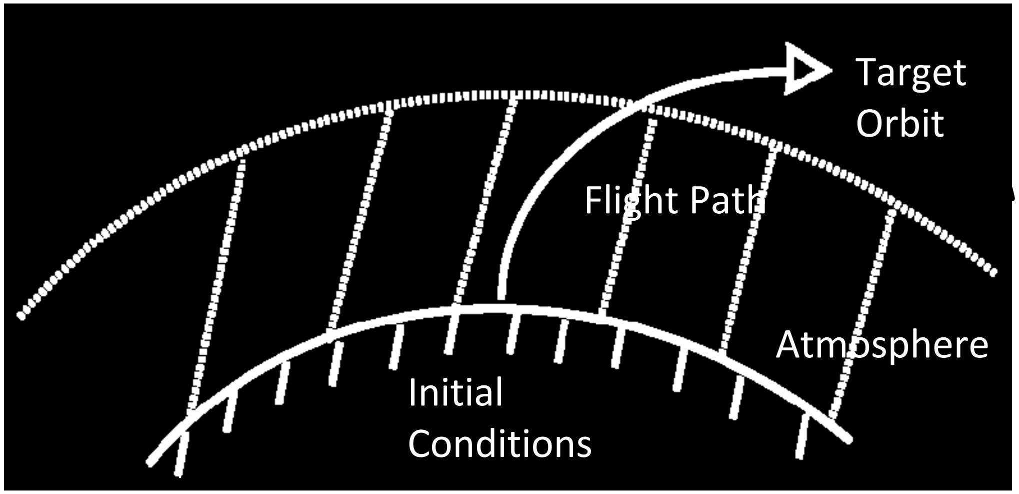
$$T = \underset{\substack{\uparrow \\ I_{sp} = f(h),\ \text{not constant}}}{I_{SP}} g_0 \dot{m}_p = -I_{SP} g_0 \underset{\substack{\uparrow \\ \text{Vehicle change in mass}}}{\dot{m}}$$

Here,
T = T(throttle, altitude) or (I_{sp})
D = D(altitude, Mach)
W = mg(altitude, time)
$\gamma = \gamma$(time) or (V)

In Practice

In practice this differential equation of motion is solved by integration because these functions tend to be non-smooth (atmosphere), discrete (engine shutdown or staging), etc.

This general class of problem is termed a 2-point boundary value problem, in which both initial and final conditions are specified



In many cases, multiple flight paths are possible and numerical optimization is used to find the most efficient path (minimize m_p for example)

Approximate Solution: The Modified Rocket Equation

Let us look at what approximations can be made:

$$T - D - mg\sin\gamma = m\frac{dV}{dt}$$

$$dV = \frac{1}{m}\left[-I_{SP}g_0\dot{m} - D - mg\sin\gamma\right]dt$$

$$\underbrace{\int_{V_i}^{V_f} dV = -g_0\int_{m_i}^{m_f} I_{SP}\frac{dm}{m}}_{\text{Looks like idealized rocket equation (except Isp}\neq\text{const)}}\underbrace{-\int_{t_i}^{t_f}\frac{D}{m}dt - \int_{t_i}^{t_f} g\sin\gamma\, dt}_{\text{Units of velocity (note negative signs)}}$$

Define an “average” I_{sp} such that:

$$-g_0\bar{I}_{SP}\int_{m_i}^{m_f}\frac{dm}{m} = -g_0\int_{m_i}^{m_f} I_{SP}\frac{dm}{m}$$

$$\Delta V = \bar{I}_{SP}g_0\ln\left(\frac{m_i}{m_f}\right) - \Delta V_{drag} - \Delta V_{grav} - \left\{\Delta V_{gimbaling(and\ other)}\right\}$$

Generally termed “modified rocket equation” where average Isp_is less than Ispvac by about 2% - 5% for most rocket systems.

ΔV Losses

Note: if we can estimate ΔV_{losses} due to drag, gravity and other effects, then we can apply the modified rocket equation to ascent performance problem as simply as the idealized rocket equation was applied to space flight

Typical ΔV losses for Earth-to-Orbit Missions

(m/s)	Rocket	Airbreather
ΔV_{grav}	1000-1500	2500-4500
ΔV_{drag}	50-150	600-1200
$\Delta V_{steering}$	30-120	50-200
Total	1080-1770	3150-5900

Gravitational effects are generally largest of velocity losses

Performance Requirements

We want an orbital velocity around 7.78 km/s (inertial)

For due East launch from KSC

$$v_i = \omega\, r \cos(\lambda)$$

$$v_i = 410 \text{ m/s}$$

So, relative velocity change required is

$$\Delta V_{reqd} \approx 7.78\text{-}0.41 \approx 7.37 \text{ km/s}$$

Including losses, our propulsion system must deliver approximately:

8.45 – 9.14 km/s	(rocket system)
10.52 – 13.27 km/s	(airbreather)

Rocket systems typically have to deliver less ΔV than airbreather systems, but their average I_{sp} is also much lower since they need to carry their own oxidizer

Example

A simple SSTO rocket is to be flown to orbit from KSC

The rocket has the following characteristics:

$I_{sp,vac} = 455$ s

Average $I_{sp} = 440$ s

Assume a trajectory with the following velocity losses:

$\Delta V_{grav} = 1.2$ km/s

$\Delta V_{drag} = 150$ m/s

$\Delta V_{steering} = 50$ m/s

Find the payload fraction and mass ratio for the idealized case and compare with our approximation of reality via the modified rocket equation (assuming a structural ratio of 10%)

Example

$$\Delta V_{\text{reqd}}\Big|_{\text{KSC->LEO}} \approx 7.37\text{km/s}$$

For idealized launch system,

$$\Delta V = \bar{I}_{SP} g_0 \ln\left(\frac{m_i}{m_f}\right)$$

7370 = 440(9.806) ln(MR) => <u>MR = 5.52</u>

$$\frac{1}{MR} = \varepsilon + (1 - \varepsilon)\pi$$

0.1812 = 0.1 + (0.9) π => <u>π = 9.0%</u>

Applying the modified rocket equation:

7370 = (440)(9.806) ln(MR) – 1200 – 150 – 50

=> <u>MR = 7.63</u>

and 1/MR = 0.1+0.9π => <u>π = 3.4%</u>