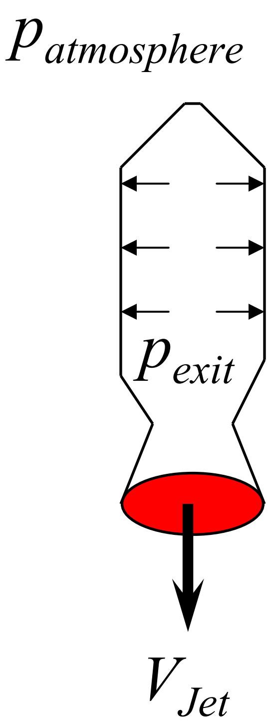
Introduction to Rocketry

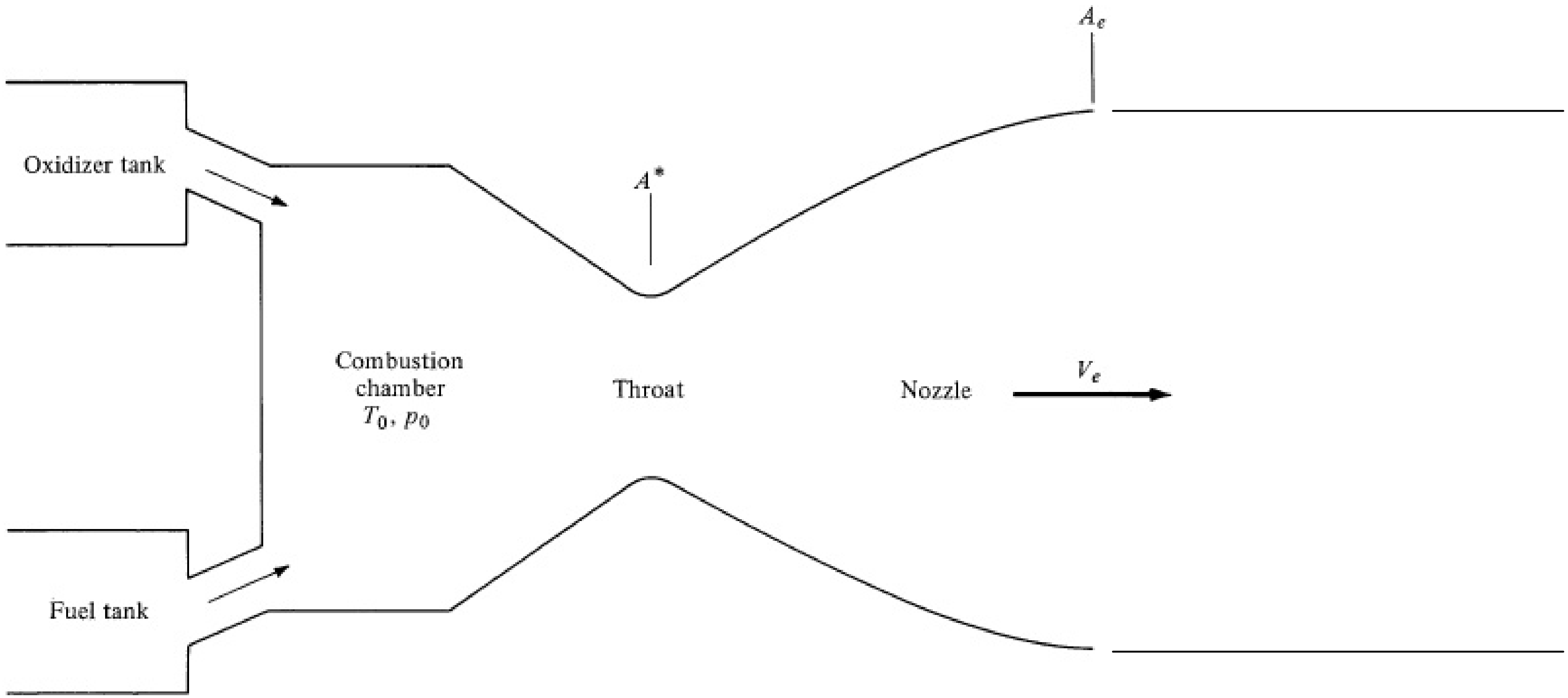
AE 1350

Rocket Motor Thrust

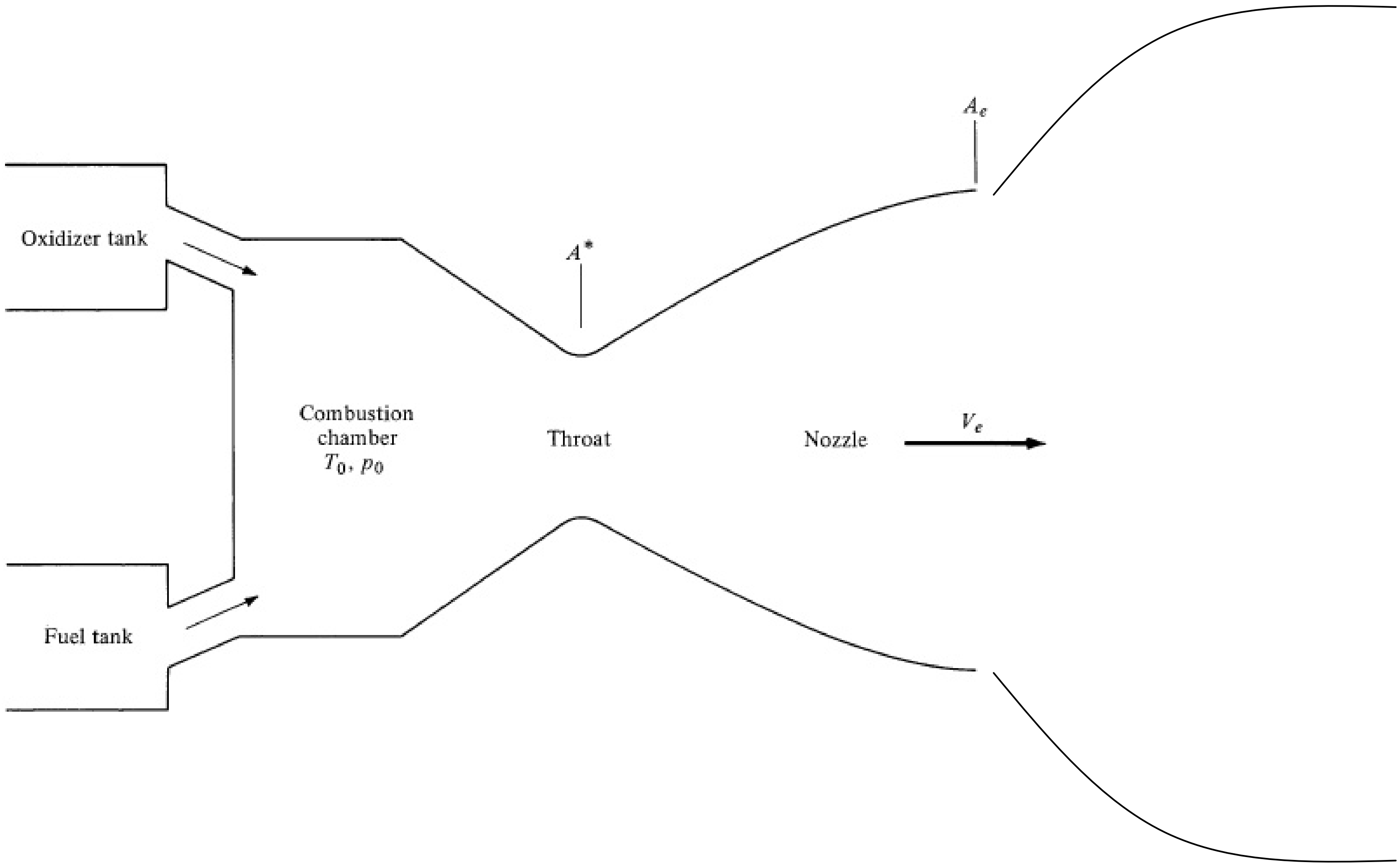


- Thrust depends on two factors:
  - Rate at which momentum leaves the rocket through the nozzle
  - Exit pressure p_{exit} and exit area A_{exit}
- $T = -dm/dt\ V_{jet} + (p_{exit} - p_{atmosphere})\ A_{exit}$
- For well designed rockets:
  - $p_{exit} = p_{atmosphere}$
  - $T = -dm/dt\ V_{jet}$
- Notice the negative sign:
  - the mass m of the rocket decreases, dm/dt is thus a negative quantity

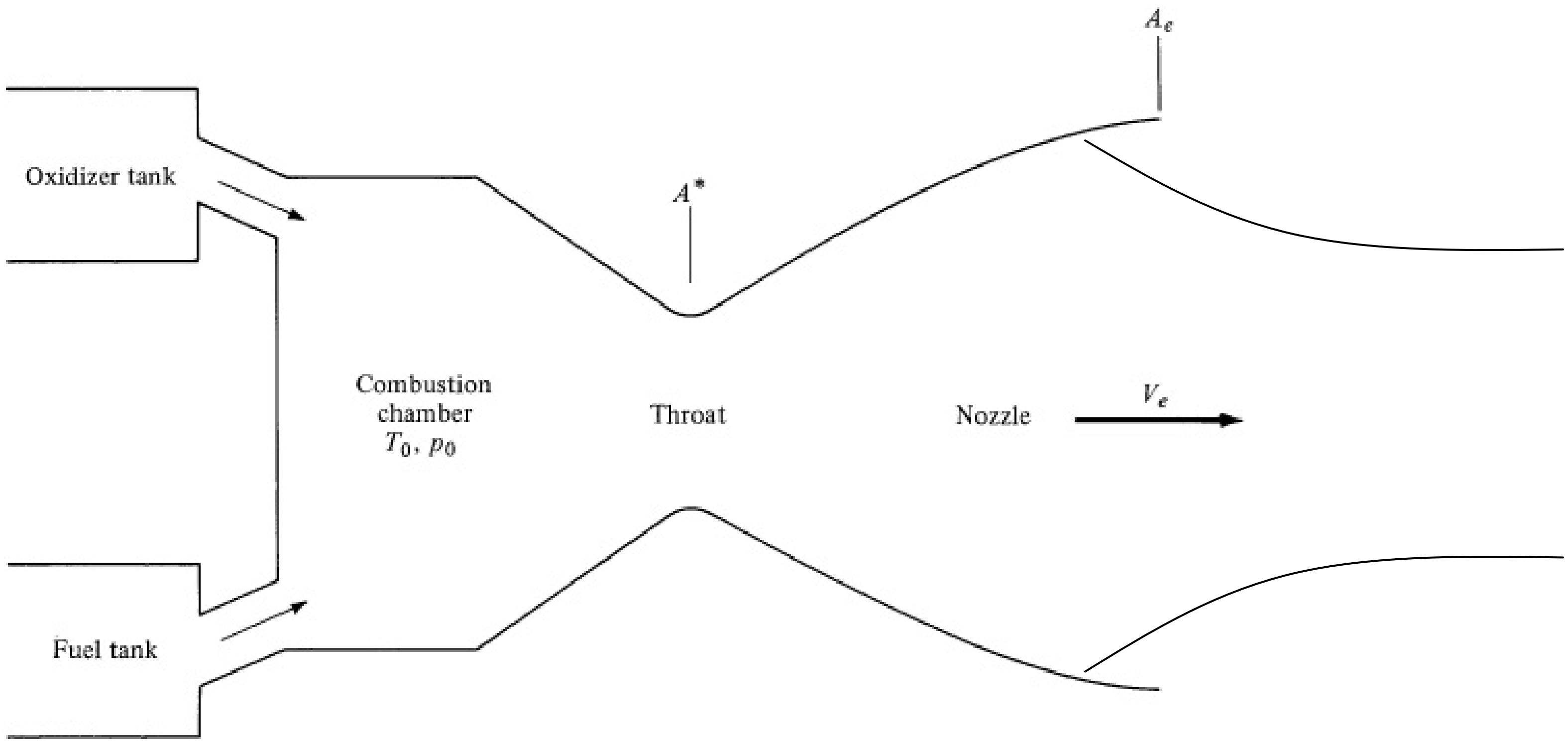
Nozzle Expansion (Ideal)



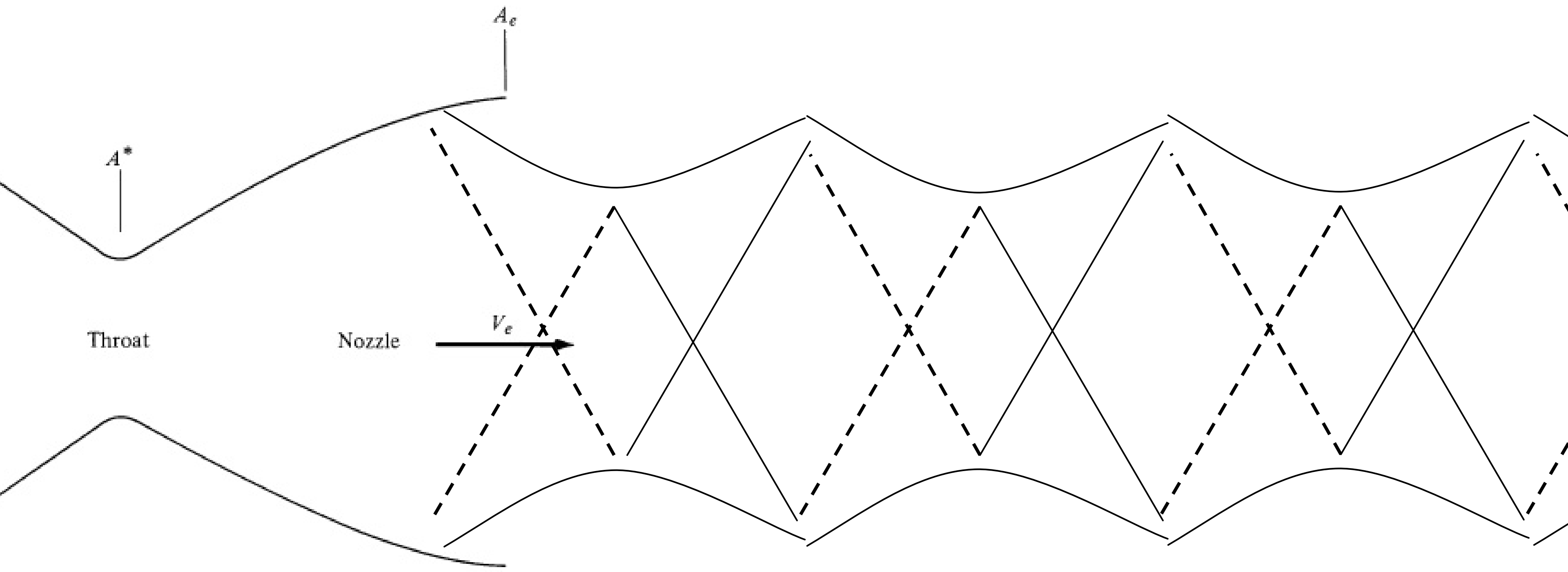
Nozzle Expansion (Under Expanded)



Nozzle Expansion (Over Expanded)



Nozzle Expansion
(Over Expanded, Supersonic Exit Velocity)



Shock Diamonds!

Nozzle Expansion (Over Expanded)

Final Velocity of the Rocket

- From the previous slide,
  Thrust: $T = - dm/dt\ V_{jet}$
- This thrust is used to accelerate the rocket, and the payload it carries
- From Newton's second law, $F = ma$
  $m\ dV/dt = T = - dm/dt\ \ V_{jet}$
- We can write the above equation as:
  $dV = - V_{jet}\ dm/m$
- Integrate:
  $\Delta V_{rocket} \ = V_{jet} \ln(m_{start}/m_{end})$
  Depends only on mass ratio, and V_{jet}

How can the final velocity of the rocket be maximized?

- From the previous slide:
  Change in the speed of the rocket (and payload it carries is given by):

$$\Delta V_{rocket} = V_{jet} \ln(m_{start}/m_{end})$$

- We must increase mass of the rocket at the start by loading it up with fuel
- We must minimize the mass of all other components
  – Structure
  – Motors
  – Avionics
  – What’s left is payload

Specific Impulse

- Specific Impulse is how long a pound of fuel can develop a pound of thrust
  $T = -I_{sp}\, dm/dt\, g$
- Thrust is Specific Impulse, multiplied by the mass flow rate of propellants and the acceleration of gravity
- We can relate I_{sp} and V_{jet}:
  $T = -dm/dt\ \ V_{jet} = -I_{sp}\, dm/dt\, g$
  $V_{jet} = I_{sp}\, g$
- So, either I_{sp} and V_{jet} are a good "figure of merit" for different rocket motor choices

Rocket Motor Options

V_{jet} (km/second)	I_{sp} (seconds)	Technology
1.6 – 2.1	170 – 220	Solid Rockets (e.g. Shuttle Solid Rocket Booster, 269 sec)
1.9 – 3.4	200 – 350	Hydrocarbon Liquid Fuel (eg. Kerosene, Saturn V main engine, 255 sec.)
4.4	455	Liquid Hydrogen and Liquid Oxygen (eg. Space Shuttle Main Engine: 453 sec.)
3.0 – 5.0	300 – 500	Nuclear Hydrogen
10 – ?	300 – ?	Electric (low thrust)

Optimizing a Rocket

- Definitions
  - m_L payload mass
  - m_p propellant mass
  - m_s structure mass
  - m_0 initial mass $= m_L + m_p + m_s$
  - m_f final mass $= m_L + m_s$

  - Payload ratio $\pi = \dfrac{m_L}{m_0}$ Structural ratio $\varepsilon = \dfrac{m_s}{m_s + m_p}$

- We have

$$\Delta V = V_{jet} \ln \frac{m_0}{m_f}$$

Optimizing a Rocket

- To simplify

$$\frac{m_f}{m_0} = \frac{m_L + m_s}{m_L + m_s + m_p} = 1 - \frac{m_p}{m_0}$$

$$= 1 - \frac{m_s + m_p}{m_0} \frac{m_p}{m_s + m_p}$$

$$= 1 - (1 - \pi)(1 - \varepsilon)$$

$$= 1 - 1 + \pi + \varepsilon - \pi\varepsilon$$

$$= \pi + (1 - \varepsilon)\pi$$

- To get change in velocity:

$$\Delta V = V_{jet} \ln\left[\frac{1}{\varepsilon - (1 - \varepsilon)\pi}\right]$$

$$= -V_{jet} \ln[\varepsilon - (1 - \varepsilon)\pi]$$

Optimal a Single-Stage Rocket

- If no payload, $\pi = 0$

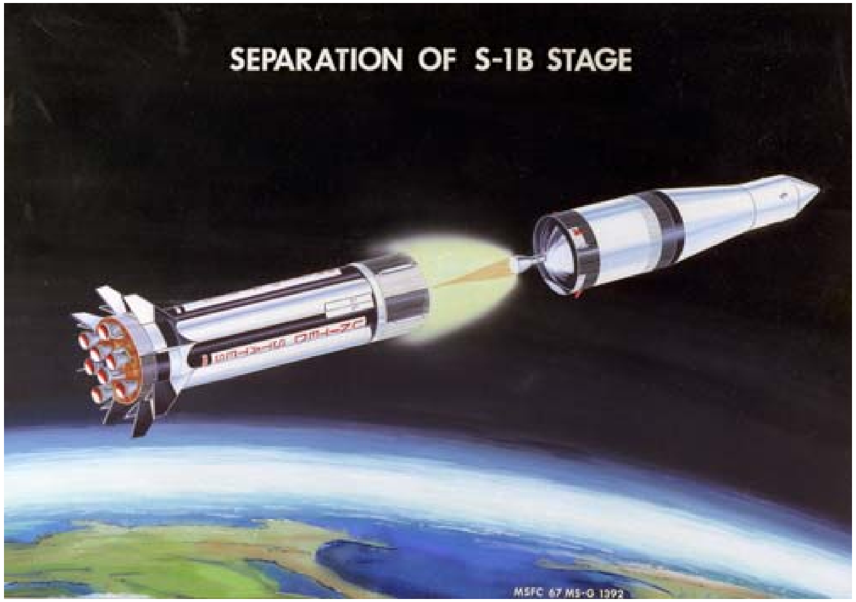
  then $\Delta V = -V_{jet} \ln \varepsilon$

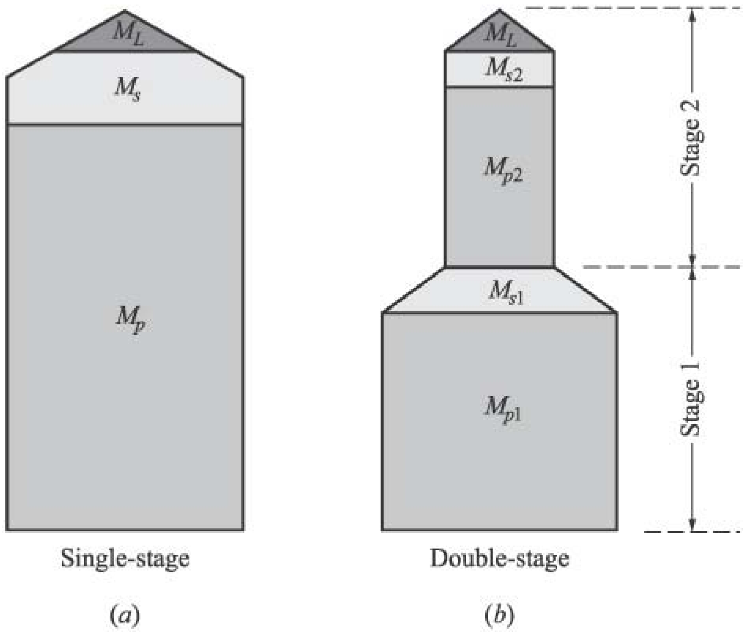
- Best ever is about $\varepsilon = 0.1$

  which creates a maximum practical speed with single stage

Single Stage vs. Multistage

- A multi-stage rocket discards earlier stages when able (fuel tanks burned, engines not needed)
- This decreases mass of the rocket and the end of later stages, and increases possible ΔV



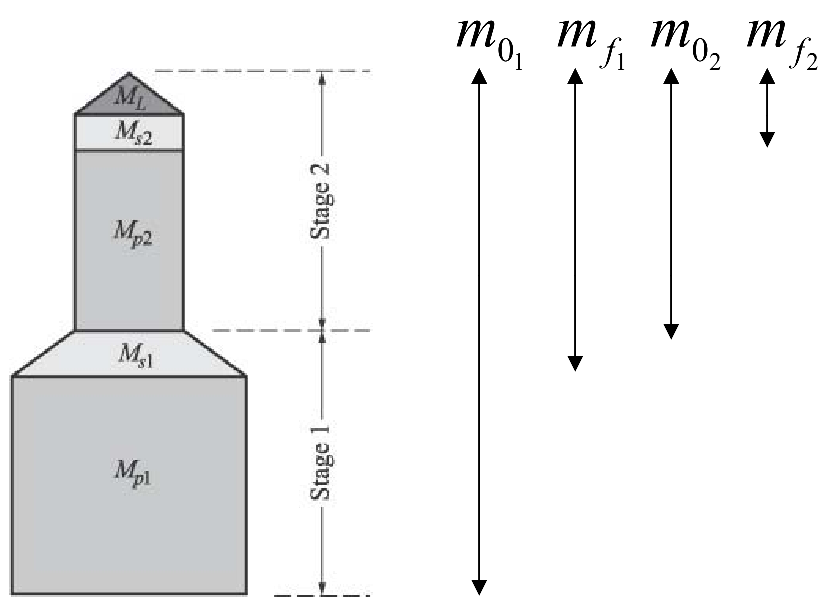


Optimum Multistage Rocket

- Define structural and "payload" factors (All future stages are payload to current stage)

$$\varepsilon_k = \frac{m_{s_k}}{m_{s_k} + m_{p_k}}, \quad \pi_k = \frac{m_{0_{k+1}}}{m_{0_k}}$$

- Overall payload factor: $\pi_* = \frac{m_L}{m_{0_1}}$



Optimum Multistage Rocket

- Total velocity change:

$$\Delta V = -\sum_{k=1}^{N} V_{jet_k} \ln\left[\varepsilon_k - \left(1-\varepsilon_k\right)\pi_k\right]$$

- Overall payload ratio is a product of individual ones:

$$\pi_* = \frac{m_L}{m_{0_1}} = \frac{m_L}{m_{0_N}} \frac{m_{0_N}}{m_{0_{N-1}}} \cdots \frac{m_{0_2}}{m_{0_1}}$$

$$= \prod_{k=1}^{N} \pi_k$$

Optimum Multistage Rocket

- Assume best values from "state of the art": V_{jet_k}, ε_k
- Pick required ΔV
- Maximize π_* (minimize initial launch mass) by changing π_k (relative size of stages):

$$\pi_* = \prod_{k=1}^{N} \pi_k$$

$$\ln \pi_* = \sum_{k=1}^{N} \ln \pi_k$$

Optimum Multistage Rocket

- Introduce Lagrange multiplier λ
- Augment cost function:

$$\ln \pi_* = \sum_{k=1}^{N} \ln \pi_k + \lambda \left\{ \frac{\Delta V}{N} + V_{jet} \ln\left[\varepsilon + (1-\varepsilon)\pi_k\right] \right\}$$

Constraint: $\Delta V = -\sum_{k=1}^{N} V_{jet} \ln\left[\varepsilon - (1-\varepsilon)\pi_k\right]$

- Take partial derivative and set equal to zero:

$$\frac{\partial \ln \pi_*}{\partial \pi_k} = \frac{1}{\pi_*} + \frac{\lambda V_{jet}(1-\varepsilon)}{\varepsilon + (1-\varepsilon)\pi_k} = 0$$

Optimum Multistage Rocket

- Solve for payload ratios:

$$\pi_k = \frac{-\varepsilon}{(1-\varepsilon)(1+\lambda V_{jet})}$$

- Need to eliminate Lagrange multiplier. Substitute into ΔV equation:

$$\Delta V = -\sum_{k=1}^{N} V_{jet} \ln\left[\varepsilon - \frac{\varepsilon}{1+\lambda V_{jet}}\right]$$

use to get λ

$$\lambda = \frac{e^{-\beta}}{V_{jet}\left(\varepsilon - e^{-\beta}\right)}, \quad \beta = \frac{\Delta V}{N V_{jet}}$$

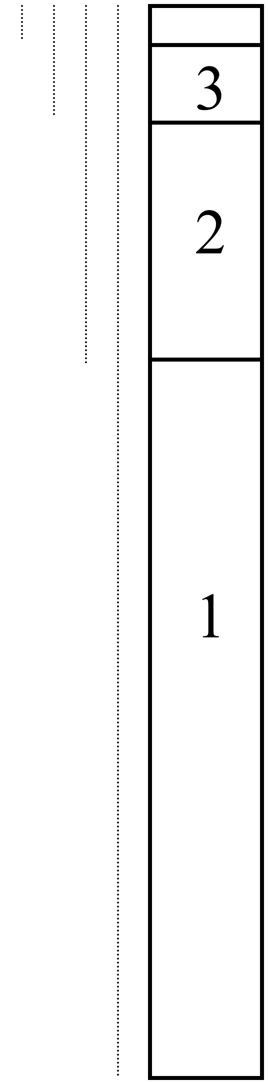
Optimum Multistage Rocket

- The final answer:

$$\pi_k = \frac{e^{-\beta}}{V_{jet}\left(\varepsilon - e^{-\beta}\right)}$$

  the same for every stage!

- Each stage smaller than predecessor, e.g. $\pi = 0.33$



Saturn V

- Thrust:
  - Stage 1: 7.7M pounds
  - Stage 2: 1.1M pounds
  - Stage 3: 225k pounds

Optimum Multistage Rocket

- Going beyond simplified version:
  - More ΔV for stages with higher I_{sp} (V_{jet})
  - Stages with higher I_{sp} (V_{jet}) should be above/after those with lower
- Saturn V
  - I_{sp} Kerosene/LOX: 304 sec (stage 1)
  - I_{sp} LOX/LH2: 421 sec (stages 2 and 3)
- Space Shuttle
  - I_{sp} solids: 269 sec (used 2 minutes)
  - I_{sp} main: 453 sec (used for entire launch)