Lecture 14
Analysis of systems with sector nonlinearities

- Sector nonlinearities
- Lur' e system
- Analysis via quadratic Lyapunov functions
- Extension to multiple nonlinearities
a function $\phi : \mathbb{R} \to \mathbb{R}$ is said to be in sector $[l, u]$ if for all $q \in \mathbb{R}$, $p = \phi(q)$ lies between $lq$ and $uq$

can be expressed as quadratic inequality

$$(p - uq)(p - lq) \leq 0 \text{ for all } q, \ p = \phi(q)$$
examples:

- sector $[-1, 1]$ means $|\phi(q)| \leq |q|
- sector $[0, \infty]$ means $\phi(q)$ and $q$ always have same sign (graph in first & third quadrants)

some equivalent statements:

- $\phi$ is in sector $[l, u]$ iff for all $q$,
  
  $$\left|\phi(q) - \frac{u + l}{2} q\right| \leq \frac{u - l}{2} |q|$$

- $\phi$ is in sector $[l, u]$ iff for each $q$ there is $\theta(q) \in [l, u]$ with $\phi(q) = \theta(q)q$
**Nonlinear feedback representation**

linear dynamical system with nonlinear feedback

\[
\begin{align*}
\dot{x} &= Ax + Bp \\
q &= Cx
\end{align*}
\]

closed-loop system: \( \dot{x} = Ax + B\phi(Cx) \)

- a common representation that separates linear and nonlinear parts
- often \( p, q \) are scalar signals
Lur’e system

A (single nonlinearity) Lur’e system has the form

\[ \dot{x} = Ax + Bp, \quad q = Cx, \quad p = \phi(q) \]

where \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) is in sector \([l, u]\)

Here \( A, B, C, l, \) and \( u \) are given; \( \phi \) is otherwise not specified.

- A common method for describing nonlinearity and/or uncertainty.

- Goal is to prove stability, or derive a bound, using only the sector information about \( \phi \).

- If we succeed, the result is strong, since it applies to a large family of nonlinear systems.
Stability analysis via quadratic Lyapunov functions

let’s try to establish global asymptotic stability of Lur’e system, using quadratic Lyapunov function \( V(z) = z^T P z \)

we’ll require \( P > 0 \) and \( \dot{V}(z) \leq -\alpha V(z) \), where \( \alpha > 0 \) is given

second condition is:

\[
\dot{V}(z) + \alpha V(z) = 2z^T P (Az + B \phi(Cz)) + \alpha z^T P z \leq 0
\]

for all \( z \) and all sector \([l, u]\) functions \( \phi \)

same as:

\[
2z^T P (Az + B p) + \alpha z^T P z \leq 0
\]

for all \( z \), and all \( p \) satisfying \((p - uq)(p - lq) \leq 0\), where \( q = Cz \)
we can express this last condition as a quadratic inequality in \((z, p)\):

\[
\begin{bmatrix}
  z \\
p
\end{bmatrix}^T
\begin{bmatrix}
  \sigma C^T C & -\nu C^T \\
-\nu C & 1
\end{bmatrix}
\begin{bmatrix}
  z \\
p
\end{bmatrix} \leq 0
\]

where \(\sigma = lu\), \(\nu = (l + u)/2\)

so \(\dot{V} + \alpha V \leq 0\) is equivalent to:

\[
\begin{bmatrix}
  z \\
p
\end{bmatrix}^T
\begin{bmatrix}
  A^T P + PA + \alpha P & PB \\
B^T P & 0
\end{bmatrix}
\begin{bmatrix}
  z \\
p
\end{bmatrix} \leq 0
\]

whenever

\[
\begin{bmatrix}
  z \\
p
\end{bmatrix}^T
\begin{bmatrix}
  \sigma C^T C & -\nu C^T \\
-\nu C & 1
\end{bmatrix}
\begin{bmatrix}
  z \\
p
\end{bmatrix} \leq 0
\]
by (lossless) S-procedure this is equivalent to: there is a \( \tau \geq 0 \) with

\[
\begin{bmatrix}
A^T P + PA + \alpha P & PB \\
B^T P & 0
\end{bmatrix} \leq \tau \begin{bmatrix}
\sigma C^T C & -\nu C^T \\
-\nu C & 1
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
A^T P + PA + \alpha P - \tau \sigma C^T C & PB + \tau \nu C^T \\
B^T P + \tau \nu C & -\tau
\end{bmatrix} \leq 0
\]

an LMI in \( P \) and \( \tau \) (2, 2 block automatically gives \( \tau \geq 0 \))

by homogeneity, we can replace condition \( P > 0 \) with \( P \geq I \)

our final LMI is

\[
\begin{bmatrix}
A^T P + PA + \alpha P - \tau \sigma C^T C & PB + \tau \nu C^T \\
B^T P + \tau \nu C & -\tau
\end{bmatrix} \leq 0, \quad P \geq I
\]

with variables \( P \) and \( \tau \)
• hence, can efficiently determine if there exists a quadratic Lyapunov function that proves stability of Lur’e system

• this LMI can also be solved via an ARE-like equation, or by a graphical method that has been known since the 1960s

• this method is more sophisticated and powerful than the 1895 approach:
  – replace nonlinearity with \( \phi(q) = \nu q \)
  – choose \( Q > 0 \) (e.g., \( Q = I \)) and solve Lyapunov equation

\[
(A + \nu BC)^T P + P(A + \nu BC) + Q = 0
\]

for \( P \)

– hope \( P \) works for nonlinear system
Multiple nonlinearities

we consider system

\[ \dot{x} = Ax + Bp, \quad q = Cx, \quad p_i = \phi_i(q_i), \quad i = 1, \ldots, m \]

where \( \phi_i : \mathbb{R} \to \mathbb{R} \) is sector \([l_i, u_i]\)

we seek \( V(z) = z^TPz \), with \( P > 0 \), so that \( \dot{V} + \alpha V \leq 0 \)

last condition equivalent to:

\[
\begin{bmatrix}
  z \\
p
\end{bmatrix}
\begin{bmatrix}
  A^TP + PA + \alpha P & PB \\
  B^TP & 0
\end{bmatrix}
\begin{bmatrix}
z \\
p
\end{bmatrix} \leq 0
\]

whenever

\[(p_i - u_i q_i)(p_i - l_i q_i) \leq 0, \quad i = 1, \ldots, m\]
we can express this last condition as

\[
\begin{bmatrix}
    z \\
p
\end{bmatrix}^T
\begin{bmatrix}
    \sigma c_i^T c_i & -\nu_i c_i^T e_i^T \\
    -\nu_i e_i c_i & e_i e_i^T
\end{bmatrix}
\begin{bmatrix}
    z \\
p
\end{bmatrix} \leq 0, \quad i = 1, \ldots, m
\]

where \( c_i \) is the \( i \)th row of \( C \), \( e_i \) is the \( i \)th unit vector, \( \sigma_i = l_i u_i \), and \( \nu_i = (l_i + u_i)/2 \).

now we use (lossy) S-procedure to get a sufficient condition: there exists \( \tau_1, \ldots, \tau_m \geq 0 \) such that

\[
\begin{bmatrix}
    A^T P + PA + \alpha P - \sum_{i=1}^{m} \tau_i \sigma_i c_i^T c_i & PB + \sum_{i=1}^{m} \tau_i \nu_i c_i^T \\
    B^T P + \sum_{i=1}^{m} \tau_i \nu_i c_i & - \sum_{i=1}^{m} \tau_i e_i e_i^T
\end{bmatrix} \leq 0
\]
we can write this as:

\[
\begin{bmatrix}
    A^T P + PA + \alpha P - C^T D F C & PB + C^T D G \\
    B^T P + D G C & -D
\end{bmatrix} \leq 0
\]

where

\[
D = \text{diag}(\tau_1, \ldots, \tau_m), \quad F = \text{diag}(\sigma_1, \ldots, \sigma_m), \quad G = \text{diag}(\nu_1, \ldots, \nu_m)
\]

- this is an LMI in variables $P$ and $D$
- $2, 2$ block automatically gives us $\tau_i \geq 0$
- by homogeneity, we can add $P \geq I$ to ensure $P > I$
- solving these LMIs allows us to (sometimes) find quadratic Lyapunov functions for Lur’e system with multiple nonlinearities (which was impossible until recently)
we consider system

\[
\begin{align*}
\dot{x}_2 &= \phi_1(x_1), \\
\dot{x}_3 &= \phi_2(x_2), \\
\dot{x}_1 &= \phi_3(-2(x_1 + x_2 + x_3))
\end{align*}
\]

where \(\phi_1, \phi_2, \phi_3\) are sector \([1 - \delta, 1 + \delta]\)

- \(\delta\) gives the percentage nonlinearity

- for \(\delta = 0\), we have (stable) linear system 

\[
\dot{x} = \begin{bmatrix}
-2 & -2 & -2 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} x
\]
let’s put system in Lur’e form:

\[ \dot{x} = Ax + Bp, \quad q = Cx, \quad p_i = \phi_i(q_i) \]

where

\[
A = 0, \quad B = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-2 & -2 & -2
\end{bmatrix}
\]

the sector limits are \( l_i = 1 - \delta, \ u_i = 1 + \delta \)

define \( \sigma = l_i u_i = 1 - \delta^2 \), and note that \( (l_i + u_i)/2 = 1 \)
we take \( x(0) = (1, 0, 0) \), and seek to bound \( J = \int_0^\infty \|x(t)\|^2 \, dt \)

(for \( \delta = 0 \) we can calculate \( J \) exactly by solving a Lyapunov equation)

we’ll use quadratic Lyapunov function \( V(z) = z^T P z \), with \( P \geq 0 \)

Lyapunov conditions for bounding \( J \): if \( \dot{V}(z) \leq -z^T z \) whenever the sector conditions are satisfied, then \( J \leq x(0)^T P x(0) = P_{11} \)

use S-procedure as above to get sufficient condition:

\[
\begin{bmatrix}
A^T P + PA + I - \sigma C^T D C & PB + C^T D \\
B^T P + DC & -D
\end{bmatrix} \leq 0
\]

which is an LMI in variables \( P \) and \( D = \text{diag}(\tau_1, \tau_2, \tau_3) \)

note that LMI gives \( \tau_i \geq 0 \) automatically
to get best bound on $J$ for given $\delta$, we solve SDP

minimize $P_{11}$

subject to

$$\begin{bmatrix}
A^T P + PA + I - \sigma C^T D C & PB + C^T D \\
B^T P + DC & -D
\end{bmatrix} \preceq 0$$

$P \succeq 0$

with variables $P$ and $D$ (which is diagonal)

optimal value gives best bound on $J$ that can be obtained from a quadratic Lyapunov function, using S-procedure
• top plot shows bound on $J$; bottom points show results for constant linear $\phi_i$'s chosen at random in interval $1 \pm \delta$

• bound is exact for $\delta = 0$; for $\delta \geq 0.15$, LMI is infeasible